Sydney Girls High School

YEAR 11

MATHEMATICS EXTENSION 1

Yearly Examination 2010

Time Allowed: **70 minutes**

Total Marks: **68**

Topics: Harder 2U, Introductory Calculus, Further Calculus, Probability, Sequences and Series, Induction and the Second Derivative and Applications of Calculus.

Instructions:

- Attempt ALL questions
- There are 4 questions, each worth 17 marks.
- Show all necessary working. Full marks may not be awarded for careless or incomplete working.
- Begin each question on a new page.
- Diagrams are NOT to scale.

Name:____________________ **Class:** ________ 9e

QUESTION 1 — Marks

a) Differentiate: $y = 3x^4 - 2x^3 + 4$ — **2**

b) In the arithmetic series $3+8+13+18+\ldots$, find:

i. the 20^{th} term. — **1**

ii. the sum of the first 20 terms. — **2**

iii. the sum of the 21^{st} to 30^{th} term. — **2**

c) A six-sided die is rolled four times. What is the probability that the number 6 does not appear in the four rolls? — **2**

d) Solve: $\frac{9}{x+4} \leq 2$ and graph the solution on a number line — **3**

e) In a geometric series, the 3^{rd} term is -8 and the 6^{th} term is 216. Find the first term and the common ratio. — **3**

f) A and B are the points (-5, 12) and (4, 9) respectively. Find the co-ordinates of the point P which divides AB externally in the ratio 5:2. — **2**

QUESTION 2 (Begin a new page) — Marks

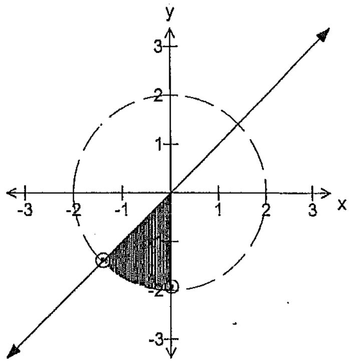
a) Evaluate: $\lim_{x \to \infty} \frac{3x^2 + 2x - 7}{6x^2 + 5}$ — **2**

b) The probability that Jack will pass a Maths test is 0.8, an English test 0.7, and a Science test 0.9. When he sits for the three tests, find the probability that Jack passes:

i. Exactly one of the three tests. — **2**

ii. At least one of the three tests. — **2**

c) Write down three inequalities which represent the shaded region below: — **3**



d) Consider the function $f(x) = 2x^3 - 9x^2 + 12x + 1$ in the domain $0 \le x \le 3$.

i. Find the stationary points and determine their nature. — **4**

ii. Find any point(s) of inflexion. — **2**

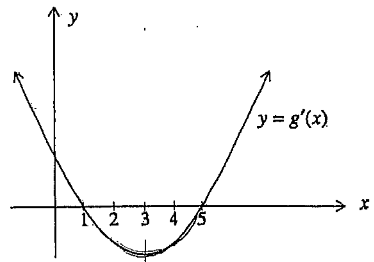
iii. Draw a sketch of the graph $y = f(x)$ in the domain $0 \le x \le 3$. — **2**

QUESTION 3 (Begin a new page) — Marks

a) Find the primitive function of $4x^2 - 3 - \frac{3}{2x^2}$. — **2**

b) Find the gradient of the normal to the curve $y = \frac{3x-5}{2x+3}$ at $x = -1$. — **3**

c)



The diagram shows the gradient function $y = g'(x)$ of the graph $y = g(x)$.

i. What features of the graph of $y = g(x)$ would you find at $x = 1$ and at $x = 5$. — **2**

ii. For what values of x is the graph $y = g(x)$ decreasing? — **2**

iii. Sketch a possible graph for $y = g(x)$ given that $y = g(x)$ passes through the origin. — **2**

d) A farmer is building a wheat silo in the shape of a cylinder, closed at both ends with radius r metres and height h metres. The silo is to be made from galvanised iron sheeting and is to have a volume of 300 cubic metres.

i. Find an expression for the height of the silo in terms of r. — **1**

ii. Show that the surface area A, of the silo is given by the equation: $A = 2\pi r^2 + \frac{600}{r}$ — **2**

iii. Hence find the minimum area of galvanised iron sheeting needed to make the silo, leaving your answer in exact form. — **3**

QUESTION 4 (Begin a new page) **Marks**

a) Differentiate: $g(x)=2x^4(3x-2)^6$. **3**
Give your answer in simplest factored form.

b) Use the principle of mathematical induction to prove that for $n\geq 1$: **4**

$$3+6+9+....+3n=\frac{3n}{2}(n+1)$$

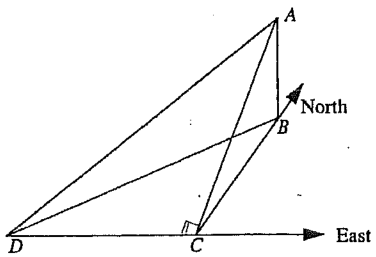
c) Kerry deposits \$150 into a superannuation fund on January 1st 2001.
She makes further deposits of \$150 on the first of each month up to and including December 1st 2010. The fund pays interest at 9% p.a., compounded at the end of each month. Find:

i. How much was in the fund on January 31st 2001. **1**

ii. How much the first deposit of \$150 is worth on December 31st 2010. **1**

iii. The total amount in the fund on December 31st 2010. **3**

d)



The angle of elevation of the top A of a building from a point C due south of it is 25°. At a second point D, which is 160 metres due west of C, the angle of elevation of the top of the building is 20°. Point B is the base of the building and on the same horizontal plane as D and C.

i. Copy and complete the diagram adding all the given information. **1**

ii. Find the height AB of the building to the nearest metre. **4**

-- END OF EXAM --

Year 11 Mathematics Extension 1

Yearly Examination 2010, SOLUTIONS

Question 1

(a) $\dfrac{dy}{dx}=12x^3-6x^2$

(b) $a=3, d=5$

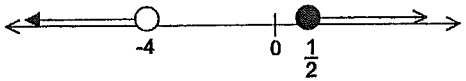
(i) $T_{20}=a+(n-1)d$
$=3+19\times5$
$=98$

(ii) $S_{20}=\dfrac{n}{2}(a+l)$
$=\dfrac{20}{2}(3+98)$
$=1010$

(iii) $S_{30}=\dfrac{n}{2}(2a+(n-1)d)$
$=\dfrac{30}{2}(2\times3+19\times5)$
$=2\,265$
$S_{30}-S_{20}=1515-1010$
$=1255$

(c) $P(\text{no six in four tosses})=\left(\dfrac{5}{6}\right)^4$
$=\dfrac{625}{1296}$

(d)



(e) $T_3=-8, T_6=216, a=?, r=?$
$ar^5=216$
$ar^2=-8$
$r^3=-27$
$r=-3$
$a=\dfrac{-8}{r^2}=-\dfrac{8}{9}$

(f) $(-5,12)\quad(4,9)$
$-5:2$
$P=\left(\dfrac{-5\times2-5\times4}{-5+2},\dfrac{-5\times9+2\times12}{-5+2}\right)$
$=(10,7)$

Question 2

(a) $\lim_{x\to\infty}\dfrac{3x^2+2x-7}{6x^2+5}$
$=\lim_{x\to\infty}\dfrac{3+\frac{2}{x}-\frac{7}{x^2}}{6+\frac{5}{x^2}}$
$=\dfrac{3}{6}=\dfrac{1}{2}$

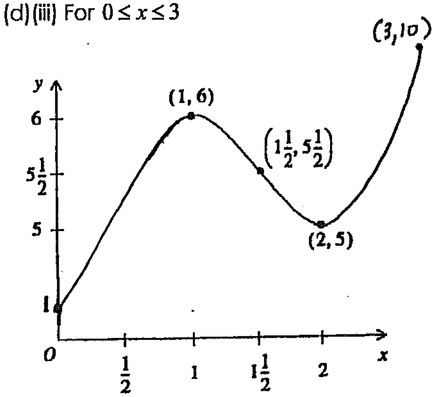
(b)

(i) $P(M\overline{E}\overline{S})+P(\overline{M}E\overline{S})+P(\overline{M}\overline{E}S)$
$=(0.8\times0.3\times0.1)+(0.2\times0.7\times0.1)$
$+(0.2\times0.3\times0.9)$
$=0.092$
$=\dfrac{23}{250}$

(ii) $P(1-\text{failing all tests})$
$=1-(0.2\times0.3\times0.1)$
$=0.994$
$=\dfrac{497}{500}$

(c) $x^2+y^2<4$
$y\le x$
$x\le 0$

(d)(iii) For $0\le x\le 3$



(d)(i) Let $y=f(x)=2x^3-9x^2+12x+1$

$\dfrac{dy}{dx}=6x^2-18x+12$

Stationary points when $\dfrac{dy}{dx}=0$

$6x^2-18x+12=0$
$6(x-1)(x-2)=0$
$x=1$ or $x=2$

When $x=1$, $y=2(1)^3-9(1)^2+12(1)+1$
$=6$

When $x=2$, $y=2(2)^3-9(2)^2+12(2)+1$
$=5$

Stationary points at $(1,6)$ and $(2,5)$.

$\dfrac{d^2y}{dx^2}=12x-18$

When $x=1$, $\dfrac{d^2y}{dx^2}=-6<0$ $\therefore$ concave down

$\therefore$ maximum turning point at $(1,6)$

When $x=2$, $\dfrac{d^2y}{dx^2}=6>0$ $\therefore$ concave up

$\therefore$ minimum turning point at $(2,5)$.

(ii) For point of inflexion $\dfrac{d^2y}{dx^2}=0$

When $12x-18=0$
$x=\dfrac{3}{2}=1\frac{1}{2}$

When $x=1\frac{1}{2}$, $y=2\left(\frac{3}{2}\right)^3-9\left(\frac{3}{2}\right)^2+12\left(\frac{3}{2}\right)+1$
$y=5\frac{1}{2}$

Sign change test:

x	1	1½	2
$\dfrac{d^2y}{dx^2}$	<0	0	>0

$\therefore$ point of inflexion is $\left(1\frac{1}{2},5\frac{1}{2}\right)$

Question 3

(a) $\frac{4x^3}{3} - 3x + \frac{3}{2x} + C$

(b) $y = \frac{3x-5}{2x+3}$

$$\frac{dy}{dx} = \frac{(2x+3)(3)-(3x-5)(2)}{(2x+3)^2}$$
$$= \frac{6x+9-6x+10}{(2x+3)^2}$$
$$= \frac{19}{(2x+3)^2}$$

Gradient of tangent $= \frac{19}{(2\times -1+3)^2} = 19$

$\therefore$ Gradient of normal $= \frac{-1}{19}$

(c) (i) **Consider $x = 1$:**

x	1^-	1	1^+
$g'(x)$	>0	0	<0

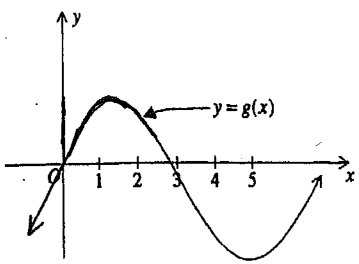
$\therefore$ at $x = 1$, $g(x)$ is a local maximum.
Consider $x = 5$.

x	5^-	5	5^+
$g'(x)$	<0	0	>0

At $x = 5$, $g(x)$ is a local minimum

(ii) The curve is decreasing for $1 < x < 5$.

(iii)



(d) (i)

$$V = \pi r^2 h$$
$$300 = \pi r^2 h$$
$$h = \frac{300}{\pi r^2}$$

(ii)

$$A = 2\pi r^2 + 2\pi r h$$
$$= 2\pi r^2 + 2\pi r \times \frac{300}{\pi r^2}$$
$$= 2\pi r^2 + \frac{600}{r}$$
$$= \frac{2\pi r^3 + 600}{r}$$

(iii)

$$\frac{dA}{dr} = 4\pi r - \frac{600}{r^2}$$
$$\frac{d^2A}{dr^2} = 4\pi + \frac{1200}{r^3}$$

When $\frac{dA}{dr} = 0$,

$$4\pi r = \frac{600}{r^2}$$
$$r^3 = \frac{150}{\pi}$$
$$r = \sqrt[3]{\frac{150}{\pi}}$$

which is a minimum since $\frac{d^2A}{dr^2} > 0$ when $r > 0$.

Question 4

(a)

$$g(x) = 2x^4(3x-2)^6$$
$$g'(x) = v\frac{du}{dx} + u\frac{dv}{dx}$$

where: $u = 2x^4$ and $v = (3x-2)^6$

$$\frac{du}{dx} = 8x^3 \qquad \frac{dv}{dx} = 6(3x-2)^5 \times 3$$
$$\frac{dv}{dx} = 18(3x-2)^5$$

$$\therefore g'(x) = (3x-2)^6 \times 8x^3 + 2x^4 \times 18(3x-2)^5$$
$$= 8x^3(3x-2)^6 + 36x^4(3x-2)^5$$
$$= 4x^3(3x-2)^5[2(3x-2)+9x]$$
$$= 4x^3(3x-2)^5[6x-4+9x]$$
$$= 4x^3(3x-2)^5(15x-4)$$

(b)

<u>Step 1: Prove true for n=1</u>

LHS = $3\times1 = 3$

RHS = $\frac{3\times1}{2}(1+1) = \frac{3}{2}\times2 = 3$

LHS = RHS

$\therefore$ True for n=1.

<u>Step 2: Assume true for n=k</u>

$$3+6+9+\ldots+3k = \frac{3k}{2}(k+1)$$

<u>Step 3: Prove true for n=k+1</u>

RTP: $3+6+9+\ldots+3k+3(k+1) = \frac{3(k+1)}{2}(k+2)$

LHS = $3+6+9+\ldots+3k+3(k+1)$

$= \frac{3k}{2}(k+1) + 3(k+1)$

$= \frac{3(k+1)}{2}(k+2)$

= RHS

$\therefore$ proven true for n=k+1

<u>Step 4:</u>

If true for n=k, then proven true for n=k+1. Proven true for n=1, and so proven true for n=2,etc. Hence by mathematical induction, statement is true for all integers n≥1.

(c)

(i) $A_1 = 150(1.0075)^1$
$= \$151.13$ (nearest cent)

(ii) $A_{120} = 150(1.0075)^{120}$
$= \$367.70$ (nearest cent)

(iii)
$A_1 = 150(1.0075)^{120}$
$A_2 = 150(1.0075)^{119}$
$A_3 = 150(1.0075)^{118}$
:
$A_{120} = 150(1.0075)^1$

∴ Superannuation

$= 150(1.0075)^{120} + 150(1.0075)^{119} + 150(1.0075)^{118} + \ldots + 150(1.0075)^1$

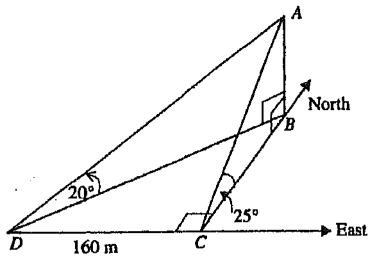
$= 150(1.0075 + 1.0075^2 + 1.0075^3 + \ldots + 1.0075^{120})$

$= 150 \times 1.0075\left[\dfrac{1.0075^{120} - 1}{1.0075 - 1}\right]$

$= 20150(1.0075^{120} - 1)$

$= \$29244.85$ (nearest cent)

(d)



In triangle ABC, $\dfrac{AB}{BC} = \tan 25°$

$BC = \dfrac{AB}{\tan 25°}$

$BC = AB\tan 65°$ ....(1)

In triangle ABD, $\dfrac{AB}{BD} = \tan 20°$

$BD = \dfrac{AB}{\tan 20°}$

$BD = AB\tan 70°$ ....(2)

In triangle BCD,
$BD^2 = BC^2 + CD^2$ ....(3)

Substitute (1) and (2) into (3):

$AB^2\tan^2 70° = AB^2\tan^2 65° + 160^2$

$AB^2(\tan^2 70° - \tan^2 65°) = 160^2$

$AB^2 = \dfrac{160^2}{\tan^2 70° - \tan^2 65°}$

$AB = 93.159987\ldots$

∴ the building is 93 m high (nearest metre).