Sydney Girls High School

LABOR OMNIA VINCIT

Mathematics Department

HSC Extension 1
Half-Yearly Examination

2004

Topics Assessed:
- *Polynomials* - *Circle Geometry* - *Inverse Trigonometric Functions* - *Integration II*

Time Allowed: 75 minutes

Instructions:

- There are 3 (THREE) questions of equal value.
- Start each question on a new page.

QUESTION 3:

a) For the function $f(x) = 2\sin^{-1} 3x$:

ii. State the domain.

iii. State the range.

iv. Sketch the graph of the function.

v. Find the equation of the tangent to the curve at the point $y = \frac{\pi}{2}$.

b) For the function $y = \frac{1}{x+2}$:

i. Find the inverse function.

ii. Find the point(s) of intersection between the function and its inverse.

c) If $f(x) = \sin^{-1} x + \cos^{-1} x, \ -1 \leq x \leq 1$:

ii. Show that $f'(x) = 0$ for all x.

iii. Show that $f(x) = \frac{\pi}{2}$ for all x.

d) Find the derivative of $y = \cos^{-1} \sqrt{1-x}$.

e) Write down the general solution for $\sin\theta = \frac{1}{\sqrt{2}}$

f) Evaluate $\int_0^{\frac{5}{3}} \frac{dx}{25+9x^2}$. Leave your answer in terms of π .

g) Evaluate, showing working, $\sin\left[\cos^{-1}\frac{4}{5} + \tan^{-1}\left(-\frac{4}{3}\right)\right]$.

END OF TEST ☺

Question 1:

$P(x) = x^3 - 7x - 6$

$P(-1) = (-1)^3 - 7\times(-1) - 6$

$= -1 + 7 - 6$

$= 0$ (2)

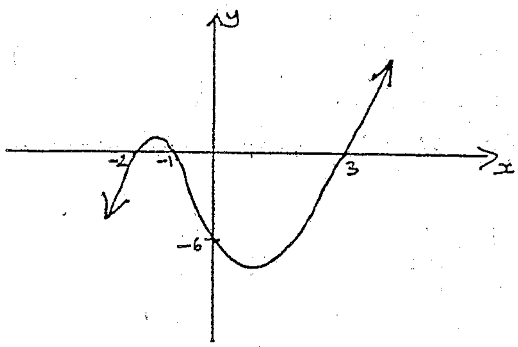
as $P(-1) = 0$ ∴ $x = -1$ is a root of $P(x)$.

$$\begin{array}{r} x^2 - x - 6 \\ x+1 \,\big)\, x^3 + 0x^2 - 7x - 6 \\ \underline{x^3 + x^2 \qquad\qquad} \\ -x^2 - 7x \quad \\ \underline{-x^2 - x \quad} \\ -6x - 6 \\ \underline{-6x - 6} \\ 0 \end{array}$$

∴ $P(x) = (x+1)(x^2 - x - 6)$

$= (x+1)(x-3)(x+2)$

∴ roots are $-1, 3$ and -2. (3)



(2)

b) $P(x) = x^3 - 4x$

$P(-3) = (-3)^3 - 4(-3)$

$= -27 + 12$

$= -15$

∴ remainder is -15. (2)

c) Let roots be $\alpha - d, \alpha, \alpha + d$. (5)

$\alpha - d + \alpha + \alpha + d = \frac{36}{8}$

$3\alpha = \frac{36}{8}$

$\alpha = 1\frac{1}{2}$

$\alpha(\alpha - d)(\alpha + d) = \frac{-21}{8}$

$\frac{3}{2}\left(\left\{\frac{3}{2}\right\}^2 - d^2\right) = \frac{-21}{8}$

$\frac{9}{4} - d^2 = -\frac{7}{4}$

$d^2 = 16$

$d = \pm 4$

∴ roots are $-2\frac{1}{2}, 1\frac{1}{2}, 5\frac{1}{2}$

d) $P(x) = x^4 - 12x + 7$ (5)

$P(0) = 0^4 - 12\times 0 + 7$

$= 7$

$P(1) = 1^4 - 12 + 7$

$= -4$

Since $P(0) > 0$ and $P(1) < 0$, the root lies between 0 and 1.

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 1 - \left(\frac{-4}{-8}\right)$

$= 1 - \frac{1}{2}$

$= 0.50$ (2 dec. pl)

$f(1) = -4$

$f'(x) = 4x^3 - 12$

$f'(1) = -8$.

∴ root is 0.50.

X is midpt of AB (given)

∴ OX ⊥ AB (line which bisects a chord from the centre of the circle is perp. to the chord)

∴ $\angle OXB = 90°$

Similarly, $\angle OYB = 90°$

$\angle OXB + \angle OYB = 90° + 90°$

$= 180°$

∴ OXBY is a cyclic quadrilateral (opp. angles are supplementary). (3)

Let $\angle ADC = x$

∴ $\angle ABY = 180° - x$ (opp. ∠s cyclic quad ABCD are supp.)

In cyclic quad OXBY

$\angle XOY = 180° - \angle ABY$ (opp ∠s cyclic quad OXBY are supp)

$= 180 - (180 - x)$

$= x$ (3)

∴ $\angle ADC = \angle XOY$.

Q2

a) $\int \frac{t}{\sqrt{1+t}}\,dt$ $u = 1+t$, $\frac{du}{dt} = 1$, $dt = du$

$\int \frac{u-1}{\sqrt{u}}\,du$

$\int (u-1)\,u^{-\frac{1}{2}}\,du$

$= \int u^{\frac{1}{2}} - u^{-\frac{1}{2}}\,du$ (5)

$= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{2}{3}\sqrt{(1+t)^3} - 2\sqrt{1+t} + c$

b) $\int_0^4 x\sqrt{16-x^2}\,dx$ $u = 16-x^2$, $\frac{du}{dx} = -2x$

$-\frac{1}{2}\int_0^4 -2x\sqrt{16-x^2}\,dx$ $du = -2x\,dx$; $x = 4 \rightarrow u = 0$; $x = 0 \rightarrow u = 16$ (5)

$\frac{1}{2}\int_0^{16} \sqrt{u}\,du$

$= \frac{1}{2}\int_0^{16} u^{\frac{1}{2}}\,du = \frac{1}{2}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{16} = \frac{1}{2}\left[\frac{2}{3}(\sqrt{16})^3\right] = \frac{64}{3} = 21\frac{1}{3}$

2c) $\int 5x\sqrt{x+4}\,dx$ $u = x+4$, $\frac{du}{dx} = 1$, $du = dx$

$5\int (u-4)(u)^{\frac{1}{2}}\,du$

$= 5\int u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\,du$

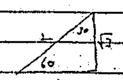
$= 5\left[\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - \frac{4u^{\frac{3}{2}}}{\frac{3}{2}}\right]$ (5)

$= 5\left[\frac{2\sqrt{(x+4)^5}}{5} - \frac{8\sqrt{(x+4)^3}}{3}\right] + c$

$= 2\sqrt{(x+4)^5} - \frac{40\sqrt{(x+4)^3}}{3} + c$

2d) $Vol = \pi\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+x^2}\,dx$

$= \pi\left[\tan^{-1} x\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$

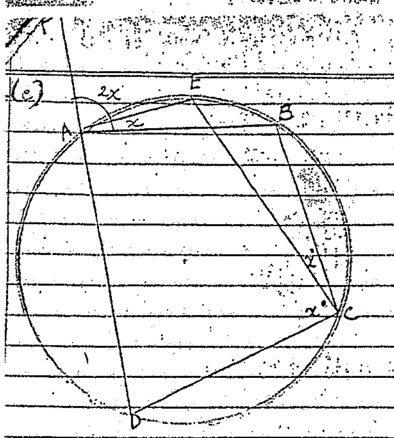


$= \pi\left[\frac{\pi}{3} - \frac{\pi}{6}\right]$ (5)

$= \pi\left[\frac{\pi}{6}\right]$

$= \frac{\pi^2}{6}\,u^3$

(e)



$\angle FAB = 2x$ ext $\angle$ of a cyclic quad ABCD

$\angle FAE = x$ ext $\angle$ of a cyclic quad AECD

$\therefore \angle EAB = 2x - x$
$= x$ (5)

EA bisects $\angle FAB$

QUESTION 3 (25 marks)

(a) $f(x) = 2\sin^{-1} 3x$

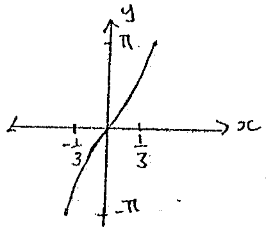
(ii) $-1 \le 3x \le 1$

D: $-\frac{1}{3} \le x \le \frac{1}{3}$ (1)

(iii) $-\frac{\pi}{2} \le \frac{y}{2} \le \frac{\pi}{2}$

R: $-\pi \le y \le \pi$ (1)

(iv)



(v) $f'(x) = 2\left[\frac{1}{\sqrt{1-u^2}} \times 3\right]$

$= \frac{6}{\sqrt{1-9x^2}}$

$= \frac{2}{\sqrt{\frac{1}{9} - x^2}}$

If $y = \frac{\pi}{2}$ then $\frac{\pi}{2} = 2\sin^{-1} 3x$

$\frac{\pi}{4} = \sin^{-1} 3x$

$\frac{1}{3\sqrt{2}} = x$

If $x = \frac{1}{3\sqrt{2}}$ $f'(\frac{1}{3\sqrt{2}}) = \frac{2}{\sqrt{\frac{1}{9} - (\frac{1}{3\sqrt{2}})^2}}$

$= \frac{2}{\sqrt{\frac{1}{9} - \frac{1}{18}}}$

$= \frac{2}{\sqrt{\frac{1}{18}}}$

$= 6\sqrt{2}$

$\therefore m = 6\sqrt{2}$

Using poi. gradient formula

$y - \frac{\pi}{2} = 6\sqrt{2}\,(x - \frac{1}{3\sqrt{2}})$

$y = 6\sqrt{2}x - 2 + \frac{\pi}{2}$ (3)

$0 = 12\sqrt{2}x - 2y + \pi - 4$

(b) $y = \frac{1}{x+2}$

(i) $y = \frac{1}{x+2}$

$x = \frac{1}{y+2}$

$y = \frac{1}{x} - 2$

$f^{-1}(x) = \frac{1}{x} - 2$ (2)

(ii) $\frac{1}{x+2} = \frac{1}{x} - 2$

$\frac{1}{x+2} = \frac{1-2x}{x}$

$x = (x+2)(1-2x)$

$x = x - 2x^2 + 2 - 4x$

$2x^2 + 4x - 2 = 0$

$x^2 + 2x - 1 = 0$

$x = \frac{-4 \pm \sqrt{32}}{4}$ (3)

$= -1 \pm \sqrt{2}$

$\therefore$ pts. of intersection are $(-1+\sqrt{2}, -1+\sqrt{2})$ & $(-1-\sqrt{2}, -1-\sqrt{2})$

(c) $f(x) = \sin^{-1} x + \cos^{-1} x$

(ii) $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$ (2)

$\therefore f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$

$= 0$

3(c)(iii) Show that $f(x) = \frac{\pi}{2}$ for all x.

Method 1: Let $a = \sin^{-1} x$

$x = \sin a$ $(-\frac{\pi}{2} \le a \le \frac{\pi}{2})$

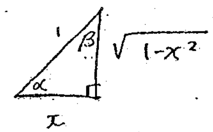
$= \cos(\frac{\pi}{2} - a)$

$\frac{\pi}{2} - a = \cos^{-1} x$

$\therefore \sin^{-1} x + \cos^{-1} x = a + \frac{\pi}{2} - a$

$= \frac{\pi}{2}$ (2)

Method 2:



Let $\beta = \sin^{-1} x$ & $\alpha = \cos^{-1} x$

$\alpha + \beta = 180° - 90°$ (angle sum of Δ)

$= 90°$

$\therefore \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

(d) $y = \cos^{-1}\sqrt{1-x}$

Let $u = (1-x)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= -\frac{1}{\sqrt{1-u^2}} \times \frac{-1}{2\sqrt{1-x}}$

$= \frac{1}{2\sqrt{1-(1-x)}} \times \frac{1}{\sqrt{1-x}}$

$= \frac{1}{2\sqrt{x}\,(\sqrt{1-x})}$

$= \frac{1}{2\sqrt{x-x^2}}$ (2)

(e) $\sin\theta = \frac{1}{\sqrt{2}}$

$\sin^{-1}(\frac{1}{\sqrt{2}}) = \frac{\pi}{4}$

for general solution

$\theta = n\pi + (-1)^n \sin^{-1} b$

$= n\pi + (-1)^n \frac{\pi}{4}$ (2)

(f) $\int_0^{\frac{5}{3}} \frac{dx}{25 + 9x^2}$

$= \frac{1}{9}\int_0^{\frac{5}{3}} \frac{dx}{\frac{25}{9} + x^2}$

$= \frac{1}{9} \cdot \frac{3}{5}\left[\tan^{-1} \frac{3x}{5}\right]_0^{\frac{5}{3}}$

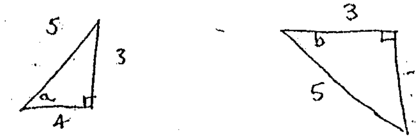
$= \frac{1}{15}\left[\tan^{-1} 1 - \tan^{-1} 0\right]$

$= \frac{\pi}{60}$ (3)

(g) $\sin^{-1}\left[\cos^{-1}\frac{4}{5} + \tan^{-1}(-\frac{4}{3})\right]$

Let $a = \cos^{-1}\frac{4}{5}$ & $b = \tan^{-1}(-\frac{4}{3})$

$\therefore \cos a = \frac{4}{5}$ $\tan b = -\frac{4}{3}$



$\sin(a+b) = \sin a \cos b + \cos a \sin b$

$= \frac{3}{5} \times \frac{3}{5} + \frac{4}{5} \times -\frac{4}{5}$

$= -\frac{7}{25}$ (2)