SYDNEY GIRLS HIGH SCHOOL

Yearly Examination

September 2004

MATHEMATICS Extension 1

Year 11

Time allowed: 75 minutes

Topics: Co-ordinate Geometry, Sequences & Series, Calculus, Harder Algebra

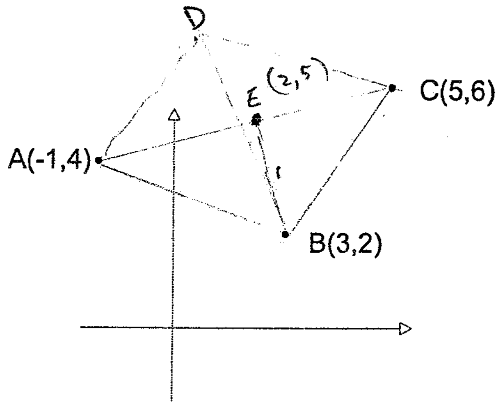
DIRECTIONS TO CANDIDATES:

- Attempt all questions
- Questions are of equal value
- There are 4 questions
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work
- Board approved calculators may be used
- Write on one side of the paper only

Year 11 Extension One Yearly Exam

Question 1:

a) Consider the points A(-1,4), B(3,2) and C (5,6)



i) Find the gradients of AB and BC and explain why $\angle ABC$ is 90^0

ii) Find the midpoint of AC, calling this point E

iii) Find the equation of the circle, centre E and radius EB

iv) If ABCD is a parallelogram, find the co-ordinates of point D

b) Find the equation of the line passing through the point of intersection of $3x - 2y + 5 = 0$ and $4x - 5y + 2 = 0$ and the point (2,3)

c) In what ratio does the point (0, 1) divide the line A(3,-5) and B (-5, 11) ?

d) Find the distance between the parallel lines $4x + 5y - 10 = 0$ and $4x + 5y - 14 = 0$

Question 2

a) i) Show that 117, 106, ~~91~~ 95 is an arithmetic progression and hence find
   ii) the 11th term
   iii) the sum of the first 11 terms
   iv) the value of the first negative term

~~b) If 2x, 2x+3 and x-1 are in geometric progression,~~
   ~~i) find a positive value of x~~
   ~~ii) the 6th term~~
   ~~iii) the sum to infinity~~

c) In an AP, the sum of the first three terms is 24 and the 5th term is 23. Find the sum of the first 8 terms

d) A couple owe \$600,000 on a house and repay \$3500 a month. If the rate of interest is 6% per annum and repayments (R) are made monthly on a reducible basis find:

   i) the amount owing after one month
   ii) the amount owing after two months
   iii) an expression for the amount owing after n months
   iv) how much is still owing after 20 years
   v) how much interest has been paid in the first 20 years of the loan

Question 3:

a) Find $\frac{dy}{dx}$ in the following:

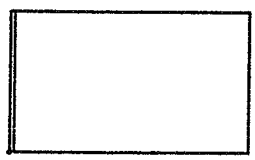
i) $y = 2x^5 - 3x^3 - 2$ ii) $y = x\sqrt{x}$ iii) $y = \frac{1}{2x^4}$

iv) $y = \sqrt{2x - 5}$ v) $y = \frac{3x-1}{x+2}$

b) Find the equation of the normal to $y = (7-2x)^4$ at the point (2,1)

c) Explain why the gradient of the curve $y = \frac{3x-1}{x+2}$ is always positive

d) A six metre long piece of wire is bent to form a rectangle with one side having two strips of wire for extra strength. Find the maximum area of the rectangle.



e) If $f(x) = x^4 - 4x + 3$,

i) find any stationary points and their nature,
ii) check and test for any inflexion points,
iii) and sketch the curve.

Question 4:

a) If f(x) = $2x^2 - x - 1$, find

i) f(x+h) in simplest expanded form

ii) the simplest expression for $\dfrac{f(x+h)-f(x)}{h}$

iii) the derivative of f(x) by first principles

b) Solve for x and sketch the solution on a number line if

$$\frac{2}{x+3} < 5$$

c) Use mathematical induction to show

$$\frac{1}{x+1}+\frac{1}{(x+1)^2}+\ldots\ldots+\frac{1}{(x+1)^n}=\frac{(x+1)^n-1}{x(x+1)^n}$$

is true for any positive integral value of n

d) Find A , B and the infinite sum of the expression:

$T_n = A\left(\dfrac{1}{2}\right)^n + B\left(\dfrac{1}{3}\right)^n$ if $T_1 = 7$ and $T_2 = 3$

end of exam

SGHS Yr 11 3Unit Yrly. Sept - 2004

1. a) i) $m_{AB} = \frac{y_2-y_1}{x_2-x_1}$ $m_{BC} = \frac{y_2-y_1}{x_2-x_1}$

$= \frac{-2}{4}$ $= \frac{4}{2}$

$= -1/2$ ✓ $= 2.$ ✓

$\therefore A\hat{B}C = 90$ since $m_{AB} \times m_{BC} = -1$ ✓

ii) $E = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$= \frac{4}{2}, \frac{10}{2}$

$= (2,5)$. ✓

(iii) $(x-h)^2 + (y-k)^2 = r^2$ $r = \sqrt{1^2+3^2}$

$(x-2)^2 + (y-5)^2 = 10.$ ✓ $r = \sqrt{10}$ ✓

$x^2 - 4x + 4 + y^2 - 10y + 25 = 10$

$x^2 - 4x + y^2 - 10y + 19 = 0$ ✓

iv). If llgram $\therefore m_{AD} \parallel m_{BC}$

& $m_{DC} \parallel m_{AB}$

Eqn AD = $(y-4) = 2(x+1)$ Eqn DC $y-6 = -1/2(x+5)$

$y - 4 = 2x + 2$ $2y - 12 = -x + 5$

① $y = 2x + 6$ ✓ ② $2y = -x + 17$

sub ① into ②

$2(2x+6) = -x + 17$

$4x + 12 = -x + 17$

$5x = 5$

$x = 1$ $\therefore y = 8$

$\therefore D(1, 8)$ ✓

Quicker to use Midpt theorem

Midpt of AC is E $(2,5)$

Let D be (a,b)

$\therefore$ Midpt of DB is $\left(\frac{a+3}{2}, \frac{b+2}{2}\right)$

$\frac{a+3}{2} = 2$ $\frac{b+2}{2} = 5$

$\therefore a = 1$ $b = 8$

b) $3x - 2y + 5 = 0$ & $4x - 5y + 2 = 0$. $(2,3)$.

$(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0$. Sub $x = 2, y = 3$

$(8 + 6 + 5) + k(8 + 15 + 2) = 0$.

$19 + 5k = 0$.

⇒ I don't think it's right

$(6 - 6 + 5) + k(8 - 15 + 2) = 0$ $k = 19$

$5 - 5k = 0 \Rightarrow k = 1$ $k = 3\frac{4}{5}$

$(3x + 2y + 5) + \frac{19}{5}(4x - 5y + 2) = 0$.

$15x + 10y + 25 + 76x - 95y + 38 = 0$

$\therefore$ Eqn. is $(3x - 2y + 5) + (4x - 5y + 2) = 0$ $91x - 85y + 63 = 0$.

$7x - 7y + 7 = 0$

Or

$x - y + 1 = 0$

$2y = 3x + 5$

$4x - \frac{5}{2}(3x+5) + 2 = 0$.

$8x - 15x + 25 + 4 = 0$.

$-21x = 7x$

$x = -3$

sub $4(-3) - 5y + 2 = 0$

$-5y = 10$

$y = -2$.

c. $(-3, 2)$

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{-2-3}{-3-2}$

$= 1$

Eqn. $(y - 2) = (x - 3)$

$= x - y - 1$

c). $0 = \frac{mx_2 + nx_1}{m+n}$ $1 = \frac{my_2 + ny_1}{m+n}$

$0 = -5m + 3n$ $m+n = 11m + 5n$

$5m = 3n.$ ✓ $= 10m - 6n = 0.$

$m=5, n=3.$ $\rightarrow$ $\frac{5m}{3n} = \frac{3n}{5n} \Rightarrow \frac{m}{n} = \frac{3}{5}$

$\therefore$ Ratio : $5:3$ ✗

d). $\underbrace{4x+5y-10=0}_{L_1}$. & $\underbrace{4x+5y-14=0}_{L_2}$.

L_2 passes through $(0, 2)$

$\therefore \frac{|4(0) + 5(2) - 14|}{\sqrt{41}}$ ✓

$= \frac{4}{\sqrt{41}}$ ✓

2. a). i) 117, 106, 95. $a = 117$, $d = -11$

$T_{11} = a + (n-1)d$

$= 117 - 110$

$= 7$ ✓

ii) $S_{11} = \frac{n}{2}(a+l)$

$= \frac{11}{2}(117+7)$

$= 682.$ ✓

iii). $T_n < 0$.

$a+(n-1)d < 0.$

$117 - 11(n-1) < 0.$

$117 < 11(n-1)$

$10.63 < (n-1)$ $\Rightarrow$ $n > 11.63...$

$n-1 = 11$ $\therefore n = 12$

$\therefore n = 12$ ✓

$T_{12} = 117 - 12 \times 11$

$= -4.$ ✓

c). $S_3 = 24$. $T_8 = 23$

$= \frac{n}{2}(2a + (n-1)d] = 24.$ $= a + (n-1)d = 23.$

$\frac{3}{2}(2a + 2d) = 24.$ $a + 4d = 23$ ✓

$2a + 2d = 16$ ✓

$\therefore 2a = 16 - 2d$

$a = 8 - d$ ✓

$\therefore 8 - d + 4d = 23$

$3d = 15$

$d = 5$ ✓

$\therefore a = 3.$ ✓

$\Rightarrow$ $\therefore S_8 = \frac{n}{2}(2a + (n-1)d)$

$= 4(6 + 7 \times 5)$

$= 164.$ ✓

2. a) i) $\cancel{600000-3500}$ $PR^1 - M$ $\quad R = \frac{\frac{6}{12}}{100} = 1\frac{1}{200}.$

$= 600000 \times 1\frac{1}{200} - 3500$

$= 599500.$ ✓

ii). $A_2 = A_1 R - M$

$= (PR - M)R - M$

$= PR^2 - M(R+1)$

$A_2 = 600000 \times (1\frac{1}{200})^2 - 3500(1\frac{1}{200}+1)$

$= \$598\,997.50.$ ✓

iii) $A_3 = A_2 R - M$

$= [PR^2 - M(R+1)]R - M$

$= PR^3 - MR(R+1) - M$

$= PR^3 - M(\underbrace{R^2 + R + 1}_{GP.})$ ✓

$= PR^3 - \frac{M(R^3-1)}{R-1}$

$\therefore A_n = PR^n - \frac{M(R^n-1)}{R-1}.$ ✓

iv). $A_{\cancel{360}\,240} = PR^{\cancel{360}\,240} - \frac{M(R^{\cancel{360}\,240}-1)}{R-1}$ $\quad$ 20 years = 20x12 months ~~= 240 months.~~

$= 60000 \times (1.005)^{\cancel{360}\,240} - \frac{3500[(1.005)^{\cancel{360}\,240}-1)}{.005}$

$= \cancel{\$97\ 712.48}$ $\$368\ 979.55$

v). $I = \cancel{360M - P - A_{360}.}$
$\cancel{= \$562257.52.}$

$I = (240 \times 3500) - (600000 - 368\ 979.55)$

$= \$608\ 979.55$

3. a) i) $y = 2x^5 - 3x^3 - 2$
$y' = 10x^4 - 9x^2$ ✓

ii) $y = x\sqrt{x}$
$= x \times x^{1/2}$
$y = x^{1\frac{1}{2}}$
$y' = 1\frac{1}{2}x^{\frac{1}{2}}$
$= \frac{3\sqrt{x}}{2}$ ✓

iii) $y = \frac{1}{2x^4} \Rightarrow y = \frac{1}{2} \times x^{-4}$
$= (2x^4)^{-1}$
$2^{-1}x^{-4}$
$y' = \frac{1}{2} \times -4x^{-5}$
$y' = \frac{1}{2} \cdot -4x^{-5} = -\frac{2}{x^5}$
$= -\frac{2}{x^5}$ ✓

iv) $y = \sqrt{2x-5}$
$y = (2x-5)^{1/2}$
$y' = \frac{1}{2} \times 2(2x-5)^{-1/2}$
$= \frac{1}{\sqrt{2x-5}}$ ✓

v) $y = \frac{3x-1}{x+2}$ $\quad u = 3x-1 \quad u' = 3 \quad v = x+2 \quad v' = 1$

$y' = \frac{vu' - uv'}{v^2}$

$= \frac{3x+6-(3x-1)}{(x+2)^2}$

$= \frac{7}{(x+2)^2}$

2. b) $y = (7-2x)^4$

$y' = 4 \times -2 \cdot (7-2x)^3$

$= -8 \ (7+2x)^3$

at $x = 2$

$m = -216$ ✓ $\therefore m_1 = \frac{1}{216}$. ✓

$\therefore$ eqn: $(y-1) = \frac{1}{216}(x-2)$

$216y - 216 = x - 2$

$\underline{0 = x - 216y + 214}$. ✓

c). $y = \dfrac{3x-1}{x+2}$

$y' = \dfrac{5}{(x+2)^2}$ (proved previous).

if $x > 0$ then $y' > 0$. since $(x+2)^2 > 0$

if $x < 0$ then $y' > 0$. because dominator is squared. ✓

d)

$2x + x + 2y = 6$

$3x + 2y = 6$.

$2y = 6 - 3x$

$y = 3 - 3/2x$ ✓

Area $= xy$

$= (3 - 3/2x)x$

$= 3x - \frac{3}{2}x^2$

$= \dfrac{6x - 3x^2}{2}$

$A' = \dfrac{6 - 6x}{2} = 0$.

$x = 1$. ✓

$\therefore x = 1, y = 1\frac{1}{2}$ ✓

$\underline{\text{Max Area} = 1.5 m^2}$. ✓

e). $f(x) = x^4 - 4x + 3$

i). $f'(x) = 4x^3 - 4 = 0$.

$4x^3 = 4$

$x = 1$ ✓

$\therefore$ Stationary point at $x = (1, 0)$ ✓

	LHS	1	RHS
$f'(x)$	< 0	$= 0$	> 0

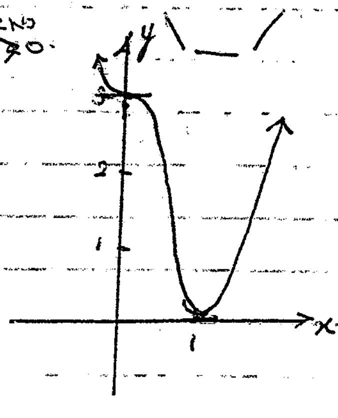
$\therefore$ $\underline{(1,0) \text{ Min Point}}$. ✓

ii). $f''(x)$ $12x^2 = 0$.

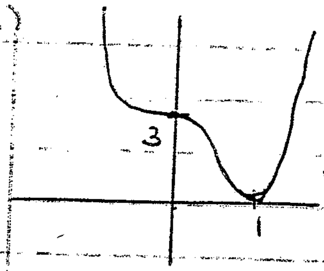
$x = 0$. ✓

	LHS	0	RHS
$f'(x)$	> 0	$= 0$	> 0

$\therefore$ (0,3) is ~~point of inflexion~~ ✓ ~~min t.p.~~



iii)



4 a) i) $f(x+h) = 2(x+h)^2 - (x+h) - 1$

$= 2x^2 + 4xh + 2h^2 - x - h - 1$

ii) $\dfrac{f(x+h) - f(x)}{h} = \dfrac{\cancel{2x^2} + 4xh + 2h^2 \cancel{-x} - h \cancel{-1} \cancel{-2x^2} \cancel{+x} \cancel{+1}}{h}$

$= \dfrac{\cancel{h}(4x + 2h - 1)}{\cancel{h}}$

iii) $\lim_{h \to 0} = 4x + 2h - 1$

$\lim_{h \to 0} = 4x - 1$

b) $\dfrac{2}{x+3} < 5$

$2(x+3) < 5(x+3)^2$

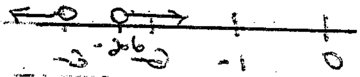
$2x + 6 < 5x^2 + 30x + 45$

$5x^2 + 28x + 39 > 0$

$\dfrac{-28 \pm \sqrt{4}}{10} > 0$

$x = -2.6 \quad x = -3$

$\therefore x < -3$ and $x > -2.6$



c) Let $n = 1$

LHS $= \dfrac{1}{(x+1)}$ RHS $= \dfrac{(x+1) - 1}{x(x+1)}$

$= \dfrac{\cancel{x}}{\cancel{x}(x+1)}$

$= \dfrac{1}{x+1}$

LHS = RHS

$\therefore$ True for $n = 1$

Assume true for $n = k$

$\therefore \dfrac{1}{x+1} + \dfrac{1}{(x+1)^2} + \dots \dfrac{1}{(x+1)^k} = \dfrac{(x+1)^k - 1}{x(x+1)^k}$

Prove. $n=k+1$

LHS : $\frac{(x+1)^k-1}{x(x+1)^k} + \frac{1}{(x+1)^{k+1}}$ RHS $= \frac{(x+1)^{k+1}-1}{x(x+1)^{k+1}}$

$= \frac{[(x+1)^k-1](x+1)}{x(x+1)^k(x+1)} + \frac{1(x)}{(x+1)^k x(x+1)(x)}$ ✓

$\frac{(x+1)(x-1)^k \cancel{-x} - 1 + x}{x(x+1)(x+1)^k}$

$= \frac{(x-1)^{k+1}-1}{x(x+1)(x+1)^k}$ ✓

$= \frac{(x-1)^{k+1}-1}{x(x+1)^{k+1}}$ ✓

LHS = RHS

∴ If true for $n=k$ and $n=k+1$, since true for $n=1$ ∴ true for All positive n.

d. $T_1 = 7 = A \times \frac{1}{2} + B \frac{1}{3}$ ①

$T_2 = 3 = A(\frac{1}{2})^2 + B(\frac{1}{3})^2$ ②

$\frac{A}{2} + \frac{B}{3} = 7$ ①

$\frac{A}{4} + \frac{B}{9} = 3$ ②

$\frac{3A+2B}{6} = 7$ ① ✓ $\frac{9A+4B}{36} = 3$

$3A + 2B = 42$

$6A + 4B = 84$ ① $9A + 4B = 108$ ②

② − ① $3A = 24$

$A = 8$.

$6(8) + 4B = 84$ ✓

$4B = 36$

$B = 9$.

∴ $T_n = 8(\frac{1}{2})^n + 9(\frac{1}{3})^n$

$T_3 = 8(\frac{1}{2})^3 + 9(\frac{1}{3})^3$

$= 1\frac{1}{3}$.

Note: $T_n = 8(\frac{1}{2})^n + 9(\frac{1}{3})^n$

$T_1 = 8(\frac{1}{2})^1 + 9(\frac{1}{3})^1$

$T_2 = 8(\frac{1}{2})^2 + 9(\frac{1}{3})^2$

$\vdots$

$T_n = 8(\frac{1}{2})^n + 9(\frac{1}{3})^n$

$T_1 = 7$, $T_2 = 3$, $T_3 = 1\frac{1}{3}$. $T_4 = \frac{11}{18}$. $T_5 = \frac{31}{108}$. $T_6 = \frac{89}{648}$.

$S_\infty = \frac{a}{1-r}$

$= \frac{7}{1-r}$

No Common Ratio, near 0.47

$S_\infty = S_1 + S_2$

$= \frac{4}{1-\frac{1}{2}} + \frac{3}{1-\frac{1}{3}}$

$= 8 + \frac{9}{2} = 12\frac{1}{2}$.

also, could say $T_1 = 7$, $T_2 = 3$.

$T_2 = ar^{n-1}$

$3 = 7 \times r^1$

$\frac{3}{7} = r$

∴ $S_\infty = \frac{a}{1-r}$

$= \frac{7}{1-3/7}$

$= 12\frac{1}{4}$.