SYDNEY GIRLS HIGH SCHOOL

LABOR OMNIA VINCIT

2000

MATHEMATICS

4 UNIT - second paper

May Assessment

Time allowed - 90 minutes

DIRECTIONS TO CANDIDATES

NAME [redacted]

- Attempt ALL questions.
- Questions are not of equal value - part marks are shown
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Each question attempted should be started on a new sheet. Write on one side of the paper only.

Question 1:

Find $\int \frac{d\theta}{1+\cos 2\theta}$ [3]

Question 2:

Find $\int \sqrt{\frac{2+x}{2-x}}.dx$ [4]

Question 3:

Find $\int \frac{dx}{\sqrt{4x^2 - 8x}}$ [5]

Question 4:

Find $\int \frac{x^3.dx}{(x+1)^2}$ [5]

Question 5:

Evaluate $\int_1^e \frac{\log_e x}{x^2}.dx$ [6]

Question 6:

Find $\int \frac{x^2.dx}{\sqrt{x^2+1}}$ [6]

Question 7:

Evaluate the following integral: $\int_3^4 \frac{6x.dx}{(x+1)^2(x-2)}$ [6]

Question 8:

Find $\int \frac{2dx}{x^3 - x^2 + x - 1}$ [6]

Question 9:

Find $\int \frac{5dx}{3 \sin x + 4 \cos x}$ [8]

Question 10:

a) Find a reduction formula for $\int \tan^n x.dx$ in terms of $\int \tan^{n-2} x.dx$ [5]

b) Evaluate $\int_0^{\frac{\pi}{4}} \tan^6 x.dx$ [3]

Question 11

Find the volume of a solid whose base is a circle of radius 3 cm and each cross section perpendicular to one of the diameter is a square. [6]

Question 12

The base of a solid is the area enclosed by $x^2 = 2y$ and $y = x$. If each cross section perpendicular to the Y axis is an equilateral triangle, find the volume of the solid. [7]

Question 13:

The area enclosed by the curve
$y = x^2 + 1$, the Y axis and the lines [redacted] and $y = 4x - 3$ is rotated about the Y axis. Find the volume that is formed. $y = x$ [7]

Question 14

The area of the curve $y = \sin x$ enclosed by the X axis, $X = 0$ and $X = \pi$ is rotated about the line $x = 2\pi$. Find the volume that is formed. [7]

Question 15:

a) Show that the area enclosed by the parabola $x^2 = 4ay$ and its latus rectum is $\frac{8a^2}{3}$

(The latus rectum is the line passing through the focus, parallel to the X axis) [3]

b) A solid is a square with sides of 6 cm and has cross sectional areas perpendicular to the diagonal is that part of a parabola enclosed by its latus rectum, the latus rectum lying on the base of the solid. Find the volume of the solid. [5]

Question 16.

A circle centre (a,b) and radius r (r<a) is rotated about the Y axis. Find the volume of the solid. [7]

4U, May 2000

1. $\int \frac{d\theta}{1+\cos 2\theta}$

$\cos 2\theta = 2\cos^2\theta - 1$

$\therefore 1+\cos 2\theta = 2\cos^2\theta$

$\therefore I = \int \frac{d\theta}{2\cos^2\theta} = \frac{1}{2}\int \sec^2\theta \, d\theta$

$= \frac{1}{2}\tan\theta + C$

2. $\int \sqrt{\frac{2+x}{2-x}} \, dx \cdot \frac{\sqrt{2+x}}{\sqrt{2+x}}$

$= \int \frac{2}{\sqrt{4-x^2}} \, dx + \int \frac{x}{\sqrt{4-x^2}} \, dx$

$= 2\sin^{-1}\frac{x}{2} - \sqrt{4-x^2} + c$

3. $\int \frac{dx}{\sqrt{4x^2-8x}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2-2x+1-1}}$

$= \frac{1}{2}\int \frac{dx}{\sqrt{(x-1)^2-1}}$

$= \frac{1}{2}\ln\left(x-1+\sqrt{x^2-2x}\right) + c$

4. $\int \frac{x^3 \, dx}{(x+1)^2}$

Now $x^2+2x+1 \,\big)\, x^3+0x^2+0x+0$ gives quotient $x-2$:

$$\begin{array}{r} x-2 \\ x^3+0x^2+0x+0 \\ x^3+2x^2+x \\ \hline -2x^2-x \\ -2x^2-4x-2 \\ \hline 3x+2 \end{array}$$

$\therefore I = \int (x-2)\,dx + \int \frac{3x+2}{x^2+2x+1} \, dx$

$= \int (x-2)\,dx + \frac{3}{2}\int \frac{2x+2}{x^2+2x+1}\,dx - \int \frac{1}{(x+1)^2}\,dx$

$= \frac{x^2}{2} - 2x + \frac{3}{2}\ln(x+1)^2 + \frac{1}{x+1} + c$

OR $= \frac{x^2}{2} - 2x + 3\ln(x+1) + \frac{1}{x+1} + c$

5. $\int_1^e \frac{\ln x}{x^2} \, dx$ Let $u = \ln x$, $dv = \frac{dx}{x^2}$; $du = \frac{dx}{x}$, $v = -\frac{1}{x}$

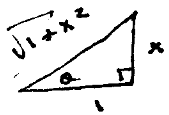
$\therefore I = \left[-\frac{1}{x}\ln x\right]_1^e + \int_1^e \frac{1}{x^2} \, dx$

$= \left[-\frac{1}{x}\ln x - \frac{1}{x}\right]_1^e$

$= \left[\frac{1}{x}\ln x + \frac{1}{x}\right]_e^1$

$= 1 - \frac{1}{e}\ln e - \frac{1}{e} = 1 - \frac{2}{e}$

6. $\int \frac{x^2 \, dx}{\sqrt{x^2+1}}$ Let $x = \tan\theta$ $\therefore dx = \sec^2\theta \, d\theta$



$I = \int \frac{\tan^2\theta \cdot \sec^2\theta \cdot d\theta}{\sqrt{\tan^2\theta+1}}$

$= \int \tan^2\theta \sec\theta \cdot d\theta$

$= \int (\sec^2\theta - 1)\sec\theta \cdot d\theta$

$= \int (\sec^3\theta - \sec\theta)\,d\theta$

For $\int \sec^3\theta \cdot d\theta$ Let $u = \sec\theta$, $dv = \sec^2\theta\,d\theta$; $du = \sec\theta\tan\theta\,d\theta$, $v = \tan\theta$

$I = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta \cdot d\theta$

$= \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\,d\theta$

$= \sec\theta\tan\theta - \int \sec^3\theta \, d\theta + \int \sec\theta \, d\theta$

$\therefore 2\int \sec^3\theta \, d\theta = \sec\theta\tan\theta + \ln(\sec\theta+\tan\theta) + c$

$\therefore \int \sec^3\theta \cdot d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln(\sec\theta+\tan\theta) + C$

$\therefore \int (\sec^3\theta - \sec\theta)\,d\theta$

$= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln(\sec\theta+\tan\theta) + C$

$= \frac{x}{2}\sqrt{1+x^2} - \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right) + C$

7. $\int_3^4 \frac{6x \, dx}{(x+1)^2(x-2)}$

Let $\frac{6x}{(x+1)^2(x-2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-2}$

$\therefore 6x = A(x+1)(x-2) + B(x-2) + C(x+1)^2$

at $x=2$, $12 = 9C$, $C = 4/3$

at $x=-1$, $-6 = -3B$, $B = 2$

Equate x^2, $0 = A+C$, $A = -4/3$

$\therefore I = \int_3^4 -\frac{4}{3}\cdot\frac{1}{x+1}\,dx + \frac{4}{3}\int_3^4 \frac{1}{x-2}\,dx + \int_3^4 \frac{2}{(x+1)^2}\,dx$

$= \left[\frac{4}{3}\ln\left(\frac{x-2}{x+1}\right) - \frac{2}{(x+1)}\right]_3^4$

$= \frac{4}{3}\ln\left(\frac{2}{5}\right) - \frac{2}{5} - \frac{4}{3}\ln\left(\frac{1}{4}\right) + \frac{1}{2}$

$= \frac{4}{3}\ln 8/5 + \frac{1}{2} - 2/5$

$= 4/3 \ln 8/5 + \frac{1}{10}$

8. $\int \frac{2\,dx}{x^3-x^2+x-1}$

$x^3-x^2+x-1 = x^2(x-1)+1(x-1)$

$= (x^2+1)(x-1)$

Let $\frac{2}{x^3-x^2+x-1} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1}$

$\therefore 2 = (Ax+B)(x-1) + C(x^2+1)$

at $x=1$, $2 = 2C$ $\therefore C = 1$

Equate x^2: $0 = A+C$, $A = -1$

at $x=0$, $2 = -B+C$, $\therefore B = -1$

$\therefore I = \int \left(\frac{1}{x-1} - \frac{x}{x^2+1} - \frac{1}{x^2+1}\right)dx$

$= \ln(x-1) - \frac{1}{2}\ln(x^2+1) - \tan^{-1}x + C$

9. $\int \frac{5\,dx}{3\sin x + 4\cos x}$ let $t = \tan\frac{x}{2}$

$\therefore dx = \frac{2dt}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$

$\therefore I = \int \frac{\frac{10dt}{1+t^2}}{\frac{6t}{1+t^2}+\frac{4-4t^2}{1+t^2}}$

$= \int \frac{5dt}{3t+2-2t^2}$

$= \int\frac{-5dt}{2t^2-3t-2} = -\int\frac{5dt}{(2t+1)(t-2)}$

let $\frac{-5}{(2t+1)(t-2)} = \frac{A}{2t+1}+\frac{B}{t-2}$

$\therefore -5 = A(t-2)+B(2t+1)$

at $t=2$, $-5=5B$, $\therefore B=-1$

at $t=-1/2$, $-5 = -2\frac{1}{2}A$, $\therefore A = 2$

$\therefore I = \int\left(\frac{2}{2t+1}-\frac{1}{t-2}\right)dt$

$= \ln\left(\frac{2t+1}{t-2}\right)+c$

$= \ln\left(\frac{2\tan^{x}/_{2}+1}{\tan^{x}/_{2}-2}\right)+c$

10. $\int\tan^n x\,dx = \int\tan^{n-2}x.\tan^2x\,dx$

$=\int\tan^{n-2}x.(\sec^2x-1)\,dx$

$=\int\tan^{n-2}x.\sec^2x\,dx - \int\tan^{n-2}x.dx$

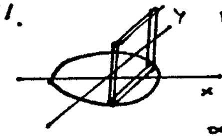
$=\frac{1}{n-1}\tan^{n-1}x - \int\tan^{n-2}x.dx$

$\therefore\int_0^{\pi/4}\tan^6x\,dx = \left[\frac15\tan^5x-\frac13\tan^3x+\tan x - x\right]_0^{\pi/4}$

$=\frac15-\frac13+1-\pi/4$

$=\frac{13}{15}-\frac{\pi}{4}$

11.



V_{slice}, $\delta V = \ell^2\delta x$

$= 4y^2\delta x$

$= 4(9-x^2)\delta x$

$V_{solid}, V = \sum_{i=0}^{\infty} 4(9-x_i^2)\delta x$

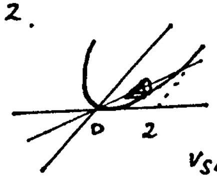
$=\int_{-3}^{3}(36-4x^2)\,dx$

$=2\int_0^3(36-4x^2)\,dx$

$=2\left[36x-\frac43x^3\right]_0^3$

$=2(108-36)$, $\therefore Vol = 144\text{ cm}^3$

12.



$y=x$

$x^2=2y$

// meet at $x=0, x=2$

$V_{slice} = \frac12 ab\sin c\,\delta y$

$a=b=x_1-x_2$

$=\sqrt{2y}-y$

$\delta V = \frac12(\sqrt{2y}-y)^2.\frac{\sqrt3}{2}\delta y$

$=\frac{\sqrt3}{4}(2y-2\sqrt2y^{3/2}+y^2)\delta y$

$V_{solid}\ V = \sum_{i=0}^{\infty}\frac{\sqrt3}{4}(2y_i-2\sqrt2y_i^{3/2}+y_i^2)\delta y$

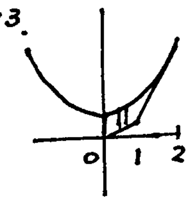
$=\frac{\sqrt3}{4}\int_0^2(2y-2\sqrt2y^{3/2}+y^2)\,dy$

$=\frac{\sqrt3}{4}.\left[y^2-\frac{2\sqrt2\cdot2}{5}y^{5/2}+\frac{y^3}{3}\right]_0^2$

$=\frac{\sqrt3}{4}\left(4-\frac{32}{5}+\frac83\right) = \frac{\sqrt3}{4}\left(\frac{60-96+40}{15}\right) = \sqrt3/15$

$\therefore$ Volume $= \sqrt3/15\ u^3$

13.



$y=x^2+1$, $y=x$, $y=4x-3$

$y=x$ & $y=4x-3$ meet at $(1,1)$

$y=x^2+1$ & $y=4x-3$ " " $(2,5)$

$\delta V_1 = \pi R^2H-\pi r^2h$

$=\pi(y_1-y_2)\{(x+\delta x)^2-x^2\}$

$=\pi(x^2+1-x)(2x\delta x$, $\delta x^2 \doteqdot 0)$

$=\pi(2x^3-2x^2+2x)\delta x$

$\delta V_2 = \pi R^2H-\pi r^2h$

$=\pi(y_1-y_2)\{(x+\delta x)^2-x^2\}$

$=\pi(x^2+1-4x+3).2x\,\delta x$

$=\pi(x^2-4x+4)2x\,\delta x$

$=\pi(x^3-4x^2+4x)\delta x$

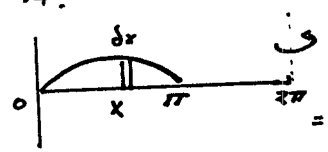
$\therefore V_1 = \pi\left(\frac{x^4}{2}-\frac{2x^3}{3}+x^2\right]_0^1$ $V_2 = 2\pi\left[\frac{x^4}{4}-\frac43x^3+\ldots\right]$

$=\pi\left(\frac12-\frac23+1\right)$ $+2\pi\left\{4-\frac{32}{3}+8-\frac14+\frac43-2\right\}$

$=\pi\left(\frac56\right)+2\pi\left(5/12\right)$

$\therefore$ Volume $= \frac{5\pi}{3}u^3$.

14.



$V_{shell} = \pi R^2H-\pi r^2H$

$=\pi y\{(2\pi-x)^2-(2\pi-x-\delta x)^2\}$

$=\pi y\{4\pi-2x\}\delta x$, $\delta x^2\doteqdot0$

$=\pi\sin x(4\pi-2x)\delta x$

$\therefore V_{solid}\ V = \sum\pi\sin x(4\pi-2x)\delta x$

$=\pi\int_0^\pi(4\pi\sin x-2x\sin x)\,dx$

for $\int2x\sin x\,dx$. Let $u=2x$ $dv=\sin x\,dx$

$du=2dx$ $v=-\cos x$

$I=-2x\cos x+\int2\cos x\,dx$

$=-2x\cos x+2\sin x+c$

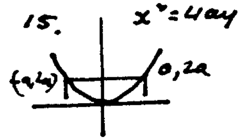
$\therefore V_{solid} = \pi\left[-4\pi\cos x+2x\cos x-2\sin x\right]_0^\pi$

$=\pi\{(4\pi-2\pi-0)-(-4\pi+0)\}$

$=\pi[2\pi+4\pi]$

$\therefore$ Volume $=6\pi^2\ u^3$

15.



$A = 4a^2 - 2\int_0^{2a}\frac{x^2}{4a}\,dx$

$=4a^2-2\left[\frac{x^3}{12a}\right]_0^{2a}$

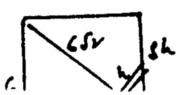
$=4a^2-\frac{1}{6a}.8a^3 = \frac{8a^2}{3}$

$V_{slice}, \delta V = \frac{8a^2}{3}\delta h$, $4a=2h$

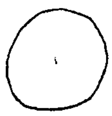
$=\frac83.\frac{h^2}{4}\delta h = \frac23h^2\delta h$

$V_{solid} = 2\int_0^{3\sqrt2}\frac23h^2\,dh$

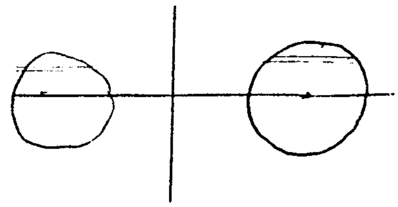
$=2\frac23.\left[\frac{h^3}{3}\right]_0^{3\sqrt2}$



$= \ldots = 24\sqrt2$



16.



$(x-a)^2 + (y-b)^2 = r^2$

Move circle to $(a, 0)$

$\therefore$ Circle is $(x-a)^2 + y^2 = r^2$

V_{slice}, $\delta V = (\pi R^2 - \pi r^2)\,\delta y$

$R = x_1$, $r = x_2$.

So $(x-a)^2 = r^2 - y^2$

$x - a = \pm\sqrt{r^2 - y^2}$

$x = a \pm \sqrt{r^2 - y^2}$

$\therefore x_1 = a + \sqrt{r^2 - y^2}$, $x_2 = a - \sqrt{r^2 - y^2}$

$\therefore \delta V = \pi\{R + r\}\{R - r\}\,\delta y$

$= \pi\{2a\}\,2\sqrt{r^2 - y^2}\,\delta y$

$= 4\pi a\sqrt{r^2 - y^2}\,\delta y$

V_{solid}, $V = \sum_{i=0}^{\infty} 4\pi a\sqrt{r^2 - y^2}\,\delta y$

$= 4\pi a \int_{-r}^{r} \sqrt{r^2 - y^2}\,dy$

$= 4\pi a \cdot \frac{1}{2}\pi r^2$ (semi-circle)

$= 2\pi^2 a r^2$