

SYDNEY GIRLS HIGH SCHOOL



2007 Assessment Task 2

Monday, 5th March, 2007

MATHEMATICS

Year 12

Time allowed: 90 minutes


Total marks: 80

Topics: Probability, Sequences & Series, Quadratic Polynomials.

DIRECTIONS TO CANDIDATES: .

- Attempt all questions
- Questions are of equal value
- There are 5 questions with part marks shown in bold
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work
- Board approved calculators may be used
- Write on one side of the paper only

Question 1 (16 marks)

- a) The first three terms of an arithmetic sequence are 5, 9, 13. 3
i. Write down a formula for the n th term.
ii. Find the eleventh term
iii. How many terms are in the series if the last term of the series is 97?
- b) The first term of an arithmetic series is 4 and the fifth term is four times the third term. 2
Find the common difference.
- c) The first two terms of an arithmetic sequence are -17, -14. 3
i. Write down the sum of the first n terms.
ii. Find the sum of the first twenty terms.
iii. What is the least value of n for which the sum of the first n terms is positive.
- d) The sum of the first n terms of an arithmetic series is given by 2
 $S_n = n(2n + 1)$.
Find an expression for the n^{th} term.
-  e) Find the sum of the first 2000 terms of the series 2
 $1 - 2 + 3 - 4 + 5 - 6 + \dots + (-1)^{n+1} \times n$. - 2000
- f) Find the number of terms in the geometric sequence 2
 $\frac{4}{243}, \frac{4}{81}, \dots, 36, 108$.
- g) The third term of a geometric series is 54, and the sixth term is 2. Find 2
i. the common ratio;
ii. the first term.

Question 2 (16 marks)

- a) The sequence is $\frac{1}{2}, 1, 2, 4, \dots$ is geometric. 2
Find the sum of the first ten terms.
Give the answer as a rational number in its lowest terms.
- b) An infinite geometric series has a first term of 8 and a limiting sum of 12. Calculate the common ratio. 2
- c) Express $0.\dot{4}\dot{7}$ as the sum of an infinite geometric progression. Hence express $0.\dot{4}\dot{7}$ as a simple fraction. 2
- d) Find the number which when added to each of 2, 6, 13 gives a set of three numbers in geometric progression. 2

- e) Rosie joins a superannuation fund by investing \$3000 at 9% p.a. compound interest at the beginning of each year for 28 years. 3
 Find the accumulated value of the investment after twenty-eight years. Write your answer correct to the nearest dollar.
- f) When Melissa left school she borrowed \$15 000 to buy her first car. The interest rate on the loan was 18% p.a. reducible. The money is to be paid back in equal monthly instalments over 5 years. At the end of each month interest is added to the principle before the monthly instalment is deducted. 5
 Let the amount of each monthly payment be M dollars and the amount owing after n payments be A_n .
- i. Write down the amount A_1 owing after one payment in terms of M .
 - ii. Show that the amount owing after two payments is

$$A_2 = 15\,000(1.015)^2 - M(1 + 1.015)$$
 - iii. Write down an expression for A_{60}
 - iv. Hence calculate the amount of each monthly instalment to the nearest dollar.

Question 3 (16 marks)

- a) Natasha has four pairs of socks, each pair a different style. 1
 If she selects two socks at random, what is the probability that they form a matching pair?
- b) Comment briefly on the following statement, giving reasons for your view : 1
 “There are twelve teams in a football competition. The probability that a particular team will win is $\frac{1}{12}$ ”.
- c) A pair of dice are thrown together at random and the numbers 1 to 6 on each die are equally likely to appear. Find the probability that 5
- i) they both show a 6.
 - ii) they show a 1 and a 6.
 - iii) at least one of them shows a 1.
 - iv) they show a total of six.
 - v) the sum of the two numbers is at least 10.

- d) One hundred tickets are to be sold in a raffle. Two different tickets are to be drawn out for first and second prizes respectively. Katie buys ten tickets. 4
 Find the probability that
- i. Katie wins first prize
 - ii. Katie wins both prizes
 - iii. Katie wins neither prize
 - iv. Katie wins at least one prize.
- e) Four metal disks numbered 1, 2, 3, 4 are placed in a bag. 3
 Two disks are selected at random and placed together on a table top to form a two digit number.
- i) Draw a tree diagram to show the possible outcomes.
 - ii) Find the probability that the number formed is 21.
 - iii) Determine the probability that the number formed is divisible by 3.
- f) On a destroyer there are two lines of defence against anti-aircraft attack. These are a surface-to-air missile and a 15 mm rapid firing gun. The probability of success in hitting an attacking aircraft with each line of defence is respectively 0.9 and 0.8. 2
 Find the probability of hitting an attacking aircraft before it penetrates both defences.



Question 4 (16 marks)

- a) An urn contains 4 black and 3 white balls. Two balls are drawn at random and placed in a hat 3
- i) Draw a probability tree to show the possible outcomes. Write the probability on each branch.
 - ii) Find the probability that the hat contains two white balls.
 - iii) Find the probability that the hat contains a white and a black ball.
- b) In a Year 12 class the probability that a student plays soccer is $P(S) = \frac{3}{4}$ and that a student plays cricket is $P(C) = \frac{1}{3}$. 2
 The probability that a student plays both Soccer and Cricket is $P(S \cap C) = \frac{1}{8}$.
 Find the probability $P(S \cup C)$ that a student selected at random from the class plays either soccer or cricket or both.
- c) In a group of 40 girls there are 29 girls who travel to school by train and 23 who travel by bus, while 7 travel by neither 3
- i) Draw a Venn diagram using the information above.
 - ii) What is the probability that a girl chosen at random travels by train and bus
 - iii) What is the probability that a girl chosen at random travels only by bus.



- d) Draw a neat sketch of $y = x^2 + 2x - 8$ showing 4
- i) x intercepts
 - ii) y intercept
 - iii) axis of symmetry
 - iv) vertex

- e) Use your graph in part d) to solve $x^2 + 2x - 8 \geq 0$ 2

- f) Find the discriminant of $2x^2 + 3x - 5$ and state whether the roots of the quadratic equation $2x^2 + 3x - 5 = 0$ are real or unreal. 2

Question 5 (16 marks)

- a) Without sketching determine whether the curve $y = 3x^2 - 4x + 5$ crosses the x -axis or not. 2

- b) Find all values of k for which the quadratic equation $kx^2 - 8x + k = 0$ has equal roots. 2

- c) For what values of m is the line $y = m(x-1)$ a tangent to the parabola $y = 2x^2$. 2

- d) The quadratic equation $2x^2 - x - 3 = 0$ has roots α and β 5
- i. calculate: $\alpha + \beta$
 - ii. calculate: $\alpha\beta$
 - iii. calculate: $\alpha^2 + \beta^2$
 - iv. calculate: $\alpha^2\beta^2$
 - v. find a quadratic equation which has roots $x = \alpha^2$ and $x = \beta^2$.

- e) The roots of the quadratic equation $mx^2 + x + n = 0$ are 2 and -1. Find m and n . 2

- f) Find the value of j such that the roots of $x^2 + 7x + j = 0$ are reciprocals of each other. 1

- g) Find the values of k if the expression $kx^2 - 12x + 3k$ is positive definite. Give exact values. 2

1 a) 5, 9, 13

$a=5, d=4$

i) $T_n = a + (n-1)d$ (1)

$= 5 + (n-1)4$

$= 5 + 4n - 4$

$T_n = 4n + 1$

ii) $T_{11} = 4 \times 11 + 1$ (1)

$= 45$

iii) $T_n = 97$

$4n + 1 = 97$ (1)

$4n = 96$

$n = 24$

b) $T_1 = 4, a = 4$

$T_5 = 4 \cdot T_3$

$a + 4d = 4(a + 2d)$

$a + 4d = 4a + 8d$

$0 = 3a + 4d$ (2)

$3 \times 4 + 4d = 0$

$4d = -12$

$d = -3$

c) i) $-17, -14$

$a = -17, d = 3$

$S_n = \frac{n}{2} [2a + (n-1)d]$

$= \frac{n}{2} [2(-17) + (n-1)3]$

$= \frac{n}{2} [-34 + 3n - 3]$ (1)

$S_n = \frac{n}{2} [3n - 37]$

ii) $S_{20} = \frac{20}{2} [3 \times 20 - 37]$

$= 10 \times 23$ (1)

$= 230$

iii) $S_n > 0$

$\frac{n}{2} (3n - 37) > 0$ (1)

~~$3n - 37 > 0$~~

$n = 13$

d) $S_n = n(2n+1) = 2n^2 + n$

$S_{n-1} = (n-1)(2(n-1)+1)$

$= (n-1)(2n-2+1)$

(2) $= (n-1)(2n-1)$

$= 2n^2 - n - 2n + 1$

$\therefore S_{n-1} = 2n^2 - 3n + 1$

$T_n = S_n - S_{n-1}$

$= 2n^2 + n - (2n^2 - 3n + 1)$

$= 2n^2 + n - 2n^2 + 3n - 1$

$\therefore T_n = 4n - 1$

e) $1 + 3 + 5 + \dots + 1999$

$= \frac{n}{2}(a+l)$ $a=1, l=1999$

$= \frac{1000}{2}(1+1999)$ $n=1000$

(2) $= 1000000$

$2 + 4 + 6 + \dots + 2000$

$= \frac{n}{2}(a+l)$ $a=2, d=2, l=2000$

$= \frac{1000}{2}(2+2000)$ $n=1000$

$= 1001000$

$Sum = 1000000 - 1001000$

$= -1000$

f) $\frac{4}{243}, \frac{4}{81}, \dots, 36, 108$

$a = \frac{4}{243}, r = 3, T_n = 108$

$T_n = a \cdot r^{n-1}$

$= \frac{4}{243} \times 3^{n-1}$

$= \frac{4}{3^5} \times 3^{n-1}$

$T_n = 4 \times 3^{n-6}$

$4 \times 3^{n-6} = 108$ (2)

$3^{n-6} = 27$

$3^{n-6} = 3^3$

$n = 9$

19) i) $T_3 = 54, T_6 = 2$
 $ar^2 = 54$
 $ar^5 = 2$
 $\frac{ar^5}{ar^2} = \frac{2}{54}$
 $r^3 = \frac{1}{27}$ (2)
 $r = \frac{1}{3}$
 ii) $a \times \left(\frac{1}{3}\right)^2 = 54$
 $a = 54 \times 9$
 $= 486$

Quest 2

a) $a = k, r = 2$
 $S_n = a \frac{(r^n - 1)}{r - 1}$
 $S_{10} = k \frac{(2^{10} - 1)}{2 - 1}$ (2)
 $= \frac{1}{2} \times (1024 - 1)$
 $= \frac{1}{2} \times 1023$
 $\therefore S_{10} = 511\frac{1}{2}$

b) $a = 8, \lim S = 12$

$\frac{a}{1-r} = 12$
 $\frac{8}{1-r} = 12$ (2)
 $8 = 12 - 12r$
 $12r = 4$

$\therefore r = \frac{1}{3}$

c) $0.4\dot{7} = 0.47 + 0.00047 + \dots$
 $+ 0.000047 + \dots$
 $= \frac{a}{1-r}, a = 0.47, r = 0.01$
 $= \frac{0.47}{1-0.01}$
 $= \frac{0.47}{0.99} = \frac{47}{99}$ (2)

d) 2, 6, 13
 G.P.: $2+x, 6+x, 13+x$
 $r = \frac{6+x}{2+x} = \frac{13+x}{6+x}$
 (2) $(6+x)^2 = (2+x)(13+x)$
 $36 + 12x + x^2 = 26 + 2x + 13x + x^2$
 $36 - 26 = 15x - 12x$
 $3x = 10$
 $x = \frac{10}{3} = 3\frac{1}{3}$

e) $P = 3000, r = 9\% \text{ pa}, n = 28$
 $= 0.09$
 $\text{Sum} = 3000 \times 1.09^{28} + 3000 \times 1.09^{27} + \dots + 3000 \times 1.09$
 $= 3000 \times (1.09 + 1.09^2 + \dots + 1.09^{28})$
 $a = 1.09$
 $= 3000 \times a \frac{(r^n - 1)}{r - 1}, r = 1.09, n = 28$
 $= 3000 \times 1.09 \times \frac{(1.09^{28} - 1)}{1.09 - 1}$
 $= 3000 \times 1.09 \times \frac{(1.09^{28} - 1)}{0.09}$ (3)
 $= \$369\ 406$

f) $P = 15000, r = 18\% \text{ pa}$
 $n = 5 \text{ yrs} = 1.5\% \text{ per month}$
 $= 60 \text{ months} = 0.015$
 i) $A_1 = 15000 \times 1.015 - M$ (1)
 ii) $A_2 = (15000 \times 1.015 - M) \times 1.015 - M$
 $= 15000 \times 1.015^2 - M \times 1.015 - M$ (1)
 $= 15000 \times 1.015^2 - M(1 + 1.015)$

26) 2 iii) $A_{60} = 15000 \times 1.015^{60} - M(1 + 1.015 + \dots + 1.015^{59})$
 $= 15000 \times 1.015^{60} - M \times \frac{(1.015^{60} - 1)}{1.015 - 1}$
 $= 15000 \times 1.015^{60} - M \times \frac{(1.015^{60} - 1)}{0.015}$ (1)

iv) $A_{60} = 0$
 $15000 \times 1.015^{60} - M \frac{(1.015^{60} - 1)}{0.015} = 0$ (2)
 $M = \frac{15000 \times 1.015^{60} \times 0.015}{(1.015^{60} - 1)}$
 $= \$225$

Question 3:

a) After the 1st sock is chosen, only one sock will complete the matching pair.

$$P(\text{matching pair}) = \frac{1}{7} \quad \textcircled{1}$$

b) Teams do not have equal ability, hence they do not have an equal chance of winning.

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c) i) $P(6 \neq 6) = \frac{1}{36} \quad \textcircled{1}$

ii) $P(1 \neq 6) = P(1,6) + P(6,1)$
 $= \frac{1}{36} + \frac{1}{36}$
 $= \frac{1}{18} \quad \textcircled{1}$

iii) $P(\text{at least one } 6) = \frac{11}{36} \quad \textcircled{1}$

iv) $P(\text{total of } 6) = \frac{5}{36} \quad \textcircled{1}$

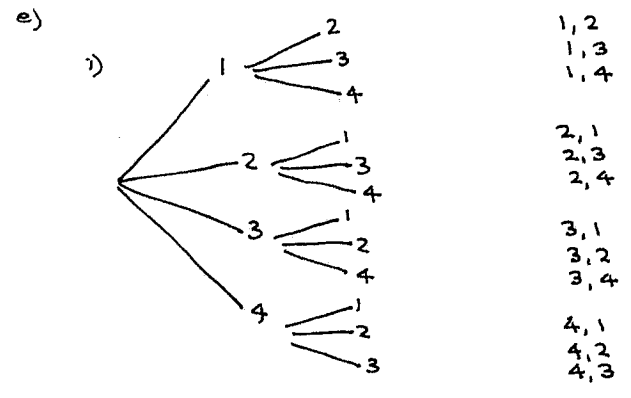
v) $P(\text{sum} \geq 10) = \frac{1}{6} \quad \textcircled{1}$

d) i) $P(\text{1st prize}) = \frac{1}{10} \quad \textcircled{1}$

ii) $P(\text{both}) = \frac{10}{100} \times \frac{9}{99}$
 $= \frac{1}{110} \quad \textcircled{1}$

iii) $P(\text{neither}) = \frac{90}{100} \times \frac{89}{99}$
 $= \frac{89}{110} \quad \textcircled{1}$

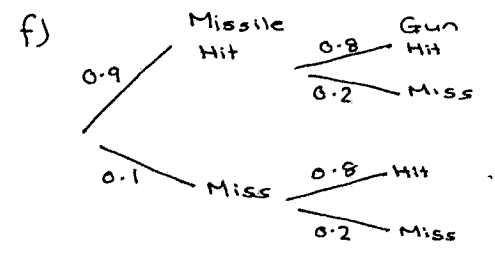
iv) $P(\text{at least } 1) = 1 - \frac{89}{110}$
 $= \frac{21}{110} \quad \textcircled{1}$



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ii) $P(21) = \frac{1}{12} \quad \textcircled{1}$

iii) $P(\text{div. by } 3) = \frac{4}{12} = \frac{1}{3} \quad \textcircled{1}$

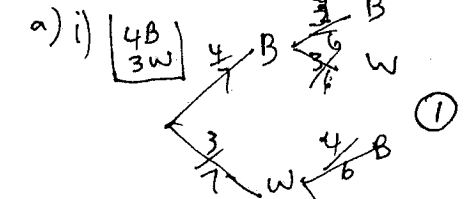


$$P(\text{hit aircraft}) = 1 - P(\text{miss both})$$

$$= 1 - (0.1 \times 0.2)$$

$$= 0.98 \quad \textcircled{2}$$

Question 4



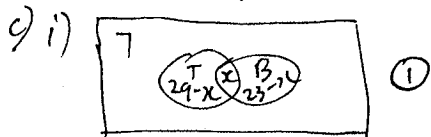
ii) $P(WW) = \frac{3}{7} \times \frac{2}{6} = \frac{1}{7} \quad \textcircled{1}$

iii) $P(WB \text{ or } BW) = \frac{4}{7} \times \frac{3}{6} + \frac{3}{7} \times \frac{4}{6} = \frac{24}{42} = \frac{4}{7} \quad \textcircled{1}$

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b) $P(S) = \frac{3}{4}, P(C) = \frac{1}{3}$
 $P(S \cap C) = \frac{1}{8}$

$P(S \cup C) = P(S) + P(C) - P(S \cap C)$
 $= \frac{3}{4} + \frac{1}{3} - \frac{1}{8}$
 $= \frac{18+8-3}{24}$
 $= \frac{23}{24}$ (2)



ii) $7 + (29-x) + (23-x) + x = 40$
 $59 - 2x = 40$

$\therefore x = 19$
 $P(\text{train} + \text{bus}) = \frac{19}{40}$ (1)

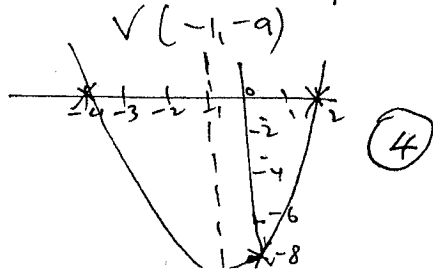
iii) $29 - x = 29 - 19 = 10$
 $P(\text{bus only}) = \frac{10}{40} = \frac{1}{4}$ (1)

d) i) $y = x^2 + 4x - 8$
 $y = (x+4)(x-2)$
 $x_{\text{int}} = -4 \text{ or } 2$

ii) $x=0, y=-8$
 $y_{\text{int}} = -8$

iii) Axis: $x = \frac{-4 \pm \sqrt{16+32}}{2}$
 $= \frac{-4 \pm \sqrt{48}}{2}$
 $\therefore x = -1$

iv) $x = -1, y = (-1)^2 + 2(-1) - 8$
 $= 1 - 2 - 8$
 $= -9$



e) $x^2 + 2x - 8 \geq 0$
 $x \leq -4 \text{ or } x \geq 2$

(2)

f) $2x^2 + 3x - 5$
 $a=2, b=3, c=-5$

$\Delta = b^2 - 4ac$
 $= 9 - 4 \times 2 \times (-5)$
 $= 9 + 40$

$\therefore \Delta = 49$ (1)
 real roots (1)

5a) $y = 3x^2 - 4x + 5$
 $a=3, b=-4, c=5$
 $\Delta = (-4)^2 - 4 \times 3 \times 5$
 $= 16 - 60$
 $= -44$ (2)

does not cross x-axis.

b) $kx^2 - 8x + k = 0$
 $a=k, b=-8, c=k$

$\Delta = b^2 - 4ac$
 $= (-8)^2 - 4 \times k \times k$
 $= 64 - 4k^2$

For equal roots $\Delta = 0$

$64 - 4k^2 = 0$
 $64 = 4k^2$
 $k^2 = 16$ (2)
 $k = \pm 4$

c) $y = m(x-1)$
 $y = 2x^2$
 $2x^2 = m(x-1)$
 $2x^2 - mx + m = 0$

$a=2, b=-m, c=m$
 $\Delta = (-m)^2 - 4 \times 2 \times m$
 $= m^2 - 8m$

Tangent, equal roots.
 $\Delta = 0$

$m^2 - 8m = 0$
 $m(m-8) = 0$
 $m = 0 \text{ or } 8$ (2)

d) $2x^2 - x + 3 = 0$
 $a=2, b=-1, c=3$

i) $\alpha + \beta = -\frac{b}{a}$
 $\therefore \alpha + \beta = \frac{1}{2}$ (1)

ii) $\alpha\beta = \frac{c}{a}$
 $\alpha\beta = \frac{3}{2}$ (1)

iii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
 $= (\frac{1}{2})^2 - 2 \times \frac{3}{2}$
 $= \frac{1}{4} - 3$
 $= -\frac{11}{4}$ (1)

iv) $\alpha^2\beta^2 = (\frac{3}{2})^2 = \frac{9}{4}$ (1)

v) eqn is $x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2 = 0$
 $x^2 - \frac{11}{4}x + \frac{9}{4} = 0$
 $4x^2 - 11x + 9 = 0$ (1)

e) $mx^2 + nx + n = 0$
 $\alpha + -1 = -\frac{b}{a}$
 $1 = \frac{-n}{m}$
 $m = -n$
 $2x - 1 = \frac{c}{a}$

$-2 = \frac{c}{m}$
 $-2 = \frac{n}{-1}$
 $n = 2$
 $\therefore m = -1, n = 2$ (2)

f) $x^2 + 7x + j = 0$
 roots $\alpha, \frac{1}{2}$

Product $\alpha \times \frac{1}{2} = \frac{c}{a}$
 $1 = \frac{j}{1}$
 $\therefore j = 1$ (1)

g) $kx^2 - 12x + 3k$
 $a=k, b=-12, c=3k$
 $\Delta = b^2 - 4ac$
 $= (-12)^2 - 4 \times k \times (3k)$
 $= 144 - 12k^2$

For unequal roots $\Delta < 0$
 $144 - 12k^2 < 0$
 $144 < 12k^2$
 $k^2 > 12$
 $\therefore k < -\sqrt{12} \text{ or } k > \sqrt{12}$

For positive definite
 $a > 0$
 $\therefore k > 0$ (2)
 $\therefore k > \sqrt{12}$