Sydney Girls High School

2004 Assessment Task 2

MATHEMATICS

Extension 2

Year 12

Time allowed - 75 minutes

Topic: Complex Numbers

Instructions

NAME 13994544

- Attempt all four questions.
- Questions are NOT of equal value.
- Total Marks = 75
- All necessary working should be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- Write on one side of the paper only.

Question One (16 marks)

1. If $z=3+4i$, $w=3-2i$:

a) Find $z+w$ [1]

b) Find $z-w$ [1]

c) Find zw [1]

d) Find $\dfrac{z}{w}$ [1]

e) Find α given $\alpha^2=z$ [3]

2 If $(1+2i)$ is a root of the quadratic equation $x^2+bx+c=0$ where b and c are real. Find b and c [2]

3. a) Solve the equation $z^7=1$ over the complex field giving your answers in mod/arg form [2]

b) Given w is the root with the smallest positive argument show that

$1+w+w^2+w^3+w^4+w^5+w^6=0$ [2]

c) Show that $\alpha=w+w^2+w^4$ and $\beta=w^3+w^5+w^6$ are the roots of the equation $z^2+z+2=0$ [3]

Question Two (22 marks)

1. Draw sketch graphs of the following loci

a) $|z-2+3i|=4$ [2]

b) $|z|<4$ and $\operatorname{Re}(z)>-2$ [2]

c) $\arg(z-i)-\arg(z+i)=\dfrac{\pi}{2}$ [2]

d) $\operatorname{Im}(z^2)=4$ [2]

e) $1\le\operatorname{Re}(z)\le 2$ and $\arg(z)\le\left|\dfrac{\pi}{3}\right|$ [2]

2. a) Sketch the locus of the complex number z where $|z-w|=\sqrt{5}$ given that $w=\dfrac{7+4i}{3-2i}$ [3]

b) Use your diagram to find maximum $|z|$ and maximum $\arg(z)$ [5]

3. Let A be the complex number $(1+i)$, B the complex number $(0+i)$.

A is rotated through π to A^*. B is rotated through $\dfrac{\pi}{2}$ to B^*.

a) Find the complex numbers representing A^* and B^* [2]

b) Hence describe the figure ABB^*A^* [2]

Question Three (16 marks)

1. Solve the following for x and y (both real) [3]

$$\frac{x+iy}{2+i}=6-i$$

2. Find the Cartesian equation of the curve represented by [4]
$|z|^2-2z-2\bar{z}=5$

3. Given that $w=\dfrac{z+4i}{z-2}$ find the locus of w if w is purely real [5]

4. Solve the following for z, $\bar{z}+\dfrac{1}{z}=2$ [4]

Question Four (21 marks)

1. Given $z=8\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ and $w=4\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ find:
   a) zw expressing you answer in the form $x+iy$ [2]
   b) $\dfrac{z}{w}$ expressing you answer in the form $x+iy$ [2]

2. Simplify $\left(1+\sqrt{3}i\right)^9$ [2]

3. By putting $z=(\cos\theta+i\sin\theta)$ and using De Moivre's Theorem
   a) Express $\cos 4\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$ [2]
   b) Express $\sin 4\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$ [1]
   c) Express $\tan 4\theta$ in terms of powers of $\tan\theta$ [2]
   d) Use your result from a) to find $\cos 4\theta$ purely in terms of powers of $\cos\theta$ [2]
   e) Using your result from part d) and by putting $x=\cos\theta$, solve the equation $8x^4-8x^2+1=0$ [4]
   f) Hence show that $\cos\frac{\pi}{8}+\cos\frac{3\pi}{8}+\cos\frac{5\pi}{8}+\cos\frac{7\pi}{8}=0$ and [2]
   g) Also that $\left(\cos\frac{\pi}{8}\right)\left(\cos\frac{3\pi}{8}\right)\left(\cos\frac{5\pi}{8}\right)\left(\cos\frac{7\pi}{8}\right)=\frac{1}{8}$ [2]

Extn 2

Solutions Task 2 2004

Q1. 1. $z = 3+4i$, $w = 3-2i$

a) $z+w = 3+4i + 3-2i$
$= 6+2i$ # ①

b) $z-w = (3+4i) - (3-2i)$
$= 6i$ # ①

c) $zw = (3+4i)(3-2i)$
$= 9+8-6i+12i$
$= 17+6i$ # ①

d) $\frac{z}{w} = \frac{3+4i}{3-2i} \times \frac{3+2i}{3+2i}$
$= \frac{9-8+12i+6i}{13}$
$= \frac{1}{13}(1+18i)$ # ①

e) let $\alpha = a+ib$
$a^2-b^2+2abi = 3+4i$
$a^2-b^2 = 3$ ① , $2ab = 4$
$(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2$
$= 25$
$a^2+b^2 = 5$
$2a^2 = 8$
$a = 2$ or $a = -2$
$b = 1$ or $b = -1$
ie $\pm(2+i)$ # ③

$\sqrt{5}$ cis

2. $(1+2i)$ is a root $\therefore$ $(1-2i)$ is a root
$\alpha+\beta = 2$, $\alpha\beta = 5$ $\Rightarrow x^2-(\alpha+\beta)x+\alpha\beta=0$
$\therefore b = -2$* , $c = 5$* ②

3. a) $z^7 = 1$
$z^7 = \text{cis}\, 0$
$z^7 = \text{cis}(0+2\pi k)$ $k = 0, 1, \ldots 7.$
take $\sqrt[7]{\ }$ and apply De Moivres Th.
$z = \text{cis}\frac{2\pi k}{7}$
$z_0 = \text{cis}\, 0$, $z_1 = \text{cis}\frac{2\pi}{7}$, $z_2 = \text{cis}\frac{4\pi}{7}$, $z_3 = \text{cis}\frac{6\pi}{7}$ #
$= 1$ $\quad z_4 = \text{cis}\frac{8\pi}{7}$, $z_5 = \text{cis}\frac{10\pi}{7}$, $z_6 = \text{cis}\frac{12\pi}{7}$ ②

b) $1+w+w^2+w^3+w^4+w^5+w^6$
Geometric $a=1$, $r=w$, $n=7$ and

its sum is given by
$S_n = \frac{a(r^n-1)}{r-1}$
$= \frac{1(w^7-1)}{w-1}$ #②
$= 0$ since $w^7 = 1$ OK sum of roots etc

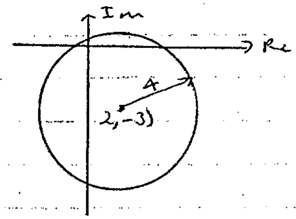
c) $\alpha+\beta = w + w^2 + w^3 + w^4 + w^5 + w^6$
$= (w+w^2+w^3+w^4+w^5+w^6+1)-1$
$= -1$

$\alpha\beta = (w+w^2+w^4)(w^3+w^5+w^6)$
$= w^4 + w^6 + w^7 + w^5 + w^7 + w^8 + w^7 + w^9 + w^{10}$
$= w^4 + w^6 + 1 + w^5 + 1 + w + 1 + w^2 + w^3$
$= 2$

Quadratic of form
$x^2 - (\alpha+\beta)x + \alpha\beta = 0$
$x^2 + x + 2 = 0$ # ③

Q2 1) a) $|z - 2 + 3i| = 4$

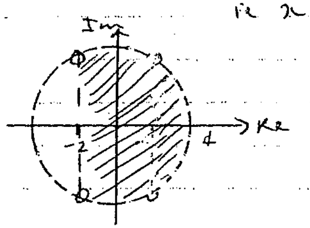
circle centre $(2, -3)$ $r = 4$



may have marked centre incorrect ②

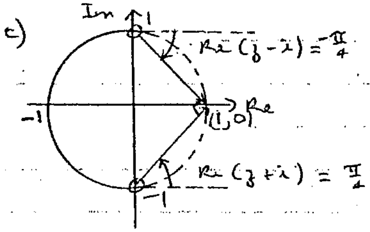
b) $|z| < 4$, $Re(z) > -2$

ie $x > -2$



②

c)



Test (1,0) RHS

$Re(z-i) - Re(z+i)$

$= -\frac{\pi}{4} - \frac{\pi}{4}$

$= -\frac{\pi}{2}$

$\therefore$ LHS of circle ②

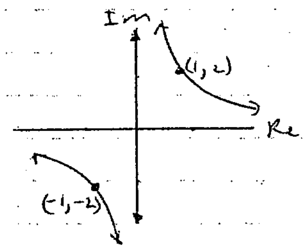
d) $Im(z^2) = 4$

ie $Im[(x+iy)^2] = 4$

$Im(x^2 - y^2 + 2xy) = 4$

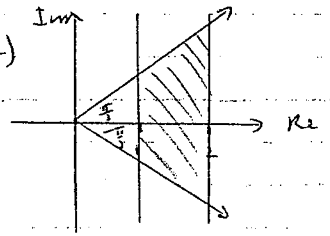
$\therefore 2xy = 4$

$xy = 2$



②

e)



$1 \leq Re(z) \leq 2$

$arg(z) \leq \left|\frac{\pi}{3}\right|$ ②

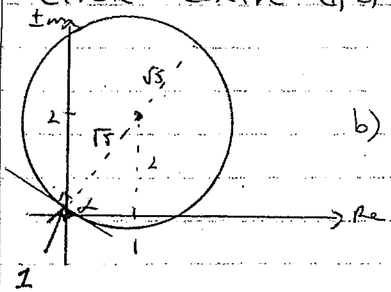
1 mark for top half only

Q2 cont

2. a) $|z - w| = \sqrt{5}$, $w = \frac{7+4i}{3-2i} \times \frac{3+2i}{3+2i}$

③ ie $|z - (1+2i)| = \sqrt{5}$ $= \frac{21 - 8 + 12i + 14i}{13}$

circle centre $(1, 2)$ $r = \sqrt{5}$ $= 1 + 2i$ 1



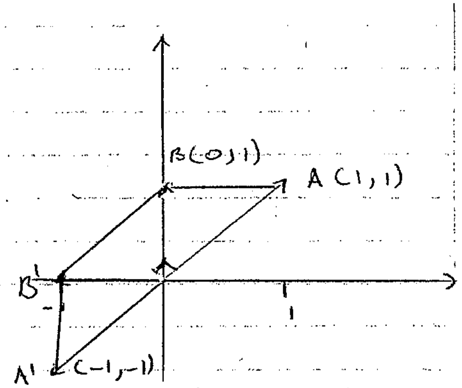
b) $Max|z| = 2\sqrt{5}$ ③

$Max\ arg(z) = \frac{\pi}{2} + \tan^{-1} 2$

or just less

$\doteqdot 153°26'$ ③

3.



a) Rotate A through π ie multiply by i^2

$(1+i)i^2 = i^2 + i^3$

$= -1 - i$ (A') ①

Rotate B through $\frac{\pi}{2}$ ie multiply by i

$(0 + i)\, i = (-1 + 0)$ ①

b) ABB'A' is an isosceles trapezium ②

Total 22

Question Three

1. $\dfrac{x+iy}{2+i} = 6-i$

$x+iy = (2+i)(6-i)$

$x+iy = 12+6i-2i-i^2$

3 $\quad x+iy = 13+4i$

$\therefore x = 13,\ y = 4.$

2. $|z|^2 - 2z - 2\bar{z} = 5$

$|z|^2 - 2(z+\bar{z}) = 5.$

Put $z = x+iy$.

then $x^2+y^2 - 2(2x) = 5.$

$x^2 - 4x + 4 + y^2 = 9.$

4 $\quad (x-2)^2 + y^2 = 3^2$

Circle: Centre $(2,0)$ Radius 3

3. $w = \dfrac{z+4i}{z-2} = \dfrac{x+iy+4i}{x+iy-2} = \dfrac{x+i(y+4)}{(x-2)+iy}$

$w = \dfrac{x+i(y+4)}{(x-2)+iy} \cdot \dfrac{(x-2)-iy}{(x-2)-iy} = \dfrac{x(x-2) - ixy + i(y+4)(x-2) - i^2y(y+4)}{(x-2)^2+y^2}.$

5 $\quad w = \dfrac{x^2 - 2x \cancel{-ixy} + \cancel{ixy} - i2y + i4x - 8i + y(y+4)}{(x-2)^2+y^2}.$

w is purely Real $\therefore$ Imaginary Part is 0.

$\therefore -2y + 4x - 8 = 0$

$\underline{2x - y = 4} \Rightarrow y = 2x-4$

excluding $(2,0)$.

4. $\bar{z} + \dfrac{1}{z} = 2.$

$\therefore \bar{z}\cdot z + 1 = 2z.$

But $\bar{z}\cdot z$ is purely real

4 $\quad$ and $\bar{z}\cdot z + 1$ is purely real

$\therefore\ 2z$ is purely real.

Hence: $z^2 + 1 = 2z \Rightarrow z^2 - 2z + 1 = 0$

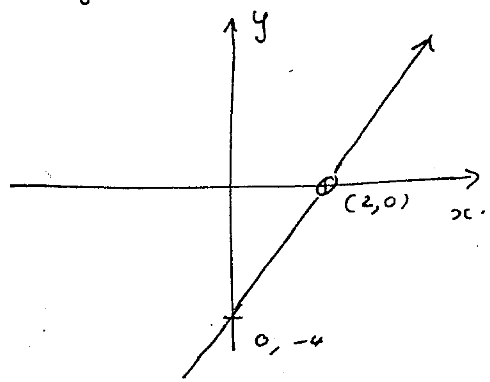
$(z-1)^2 = 0$

$\therefore \underline{\underline{z = 1}}$

3. $w = \dfrac{z+4i}{z-2}$. $\quad w$ is purely real.

then $\arg w = 0, \pi$.

i.e. $\arg\left(\dfrac{z+4i}{z-2}\right) = 0, \pi \Rightarrow \arg(z+4i) - \arg(z-2) = 0, \pi.$



Eqn $y - 0 = \dfrac{-4}{-2}(x-2)$

$y = 2(x-2)$

$\underline{\underline{y = 2x-4}}$

Excluding $(2,0)$

Question Four

1. $z = 8(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ $\quad w = 4(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

$z = 8\,\text{cis}\frac{\pi}{3}$ $\quad w = 4\,\text{cis}\frac{\pi}{6}$

OR

$z = 4 + 4\sqrt{3}i$ $\quad w = 2\sqrt{3} + 2i$

(a) $zw = (8\,\text{cis}\frac{\pi}{3})(4\,\text{cis}\frac{\pi}{6})$
$= 32\,\text{cis}(\frac{\pi}{3} + \frac{\pi}{6})$
$= 32\,\text{cis}\frac{\pi}{2}$
$= \underline{32i}$

$zw = (4 + 4\sqrt{3}i)(2\sqrt{3} + 2i)$
$= 8\sqrt{3} + 8i + 24i + 8\sqrt{3}i^2$
$= \underline{0 + 32i}$

2

(b) $\frac{z}{w} = \frac{8\,\text{cis}\frac{\pi}{3}}{4\,\text{cis}\frac{\pi}{6}}$
$= \frac{8}{4}\,\text{cis}(\frac{\pi}{3} - \frac{\pi}{6})$
$= 2\,\text{cis}\frac{\pi}{6}$
$= 2(\frac{\sqrt{3}}{2} + \frac{1}{2}i)$
$= \underline{\sqrt{3} + i}$

$\frac{z}{w} = \frac{4 + 4\sqrt{3}i}{2\sqrt{3} + 2i} = \frac{\cancel{2}(2 + 2\sqrt{3}i)}{\cancel{2}(\sqrt{3} + i)} \cdot \frac{\sqrt{3} - i}{\sqrt{3} - i}$
$= \frac{\cancel{2}(1 + \sqrt{3}i)(\sqrt{3} - i)}{\cancel{4}\ 2}$
$= \frac{\sqrt{3} - i + 3i - \sqrt{3}i^2}{2}$
$= \frac{2\sqrt{3} + 2i}{2}$
$= \underline{\sqrt{3} + i}$

2

2. $(1 + \sqrt{3}i)^9$
$z = 2(\frac{1}{2} + \frac{\sqrt{3}}{2}i)$
$z = 2\,\text{cis}\frac{\pi}{3}$

$z^9 = 2^9\,\text{cis}(\frac{\pi}{3})^9$
$z^9 = 512\ \text{cis}\ 3\pi$ (By De Moivres Thm).
$z^9 = 512\,\text{cis}\,\pi$
$z^9 = 512(-1)$
$z^9 = \underline{-512}$

2

6

3. $z = (\cos\theta + i\sin\theta)$
$z^4 = (\cos\theta + i\sin\theta)^4. = \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta + 4\cos\theta\, i^3\sin^3\theta + i^4\sin^4\theta$

By De Moivre's Thm

$\cos 4\theta + i\sin 4\theta = \cos^4\theta + 4\cos^3\theta\, i\sin\theta - 6\cos^2\theta\sin^2\theta. - 4\cos\theta\, i\sin^3\theta + \sin^4\theta.$

2 (a) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$. (Equating Real Parts

(b). $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ (Equating Imag. Parts)

1

(c). $\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta} = \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$

$\div$ top and bottom by $\cos^4\theta$

2

$\therefore \tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{\tan^4\theta - 6\tan^2\theta + 1}$

(d). $\cos 4\theta = \cos^4\theta - 6\cos^2\theta(1 - \cos^2\theta) + (1 - \cos^2\theta)^2$
$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta.$
$= 8\cos^4\theta - 8\cos^2\theta + 1$

2

(e). By putting $x = \cos\theta$
then the solutions to $8x^4 - 8x^2 + 1 = 0$
are the solutions to $\cos 4\theta = 0$
i.e. $\cos 4\theta = 0 \Rightarrow 4\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\cos 4\theta = \cos\frac{\pi}{2}, \cos\frac{3\pi}{2}, \cos\frac{5\pi}{2}, \cos\frac{7\pi}{2}$
$\Rightarrow \theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$

4

Hence solutions to $8x^4 - 8x^2 + 1 = 0$
are $x = \cos\frac{\pi}{8}, x = \cos\frac{3\pi}{8}, x = \cos\frac{5\pi}{8}, x = \cos\frac{7\pi}{8}$.

(f) Sum of Roots $\frac{-b}{a} = \frac{0}{8} = 0$
$\therefore \cos\frac{\pi}{8} + \cos\frac{3\pi}{8} + \cos\frac{5\pi}{8} + \cos\frac{7\pi}{8} = 0$

2

(g) Product of roots $\frac{e}{a} = \frac{1}{8}$

2

$\therefore \cos\frac{\pi}{8} \cdot \cos\frac{3\pi}{8} \cdot \cos\frac{5\pi}{8} \cdot \cos\frac{7\pi}{8} = \frac{1}{8}$

15