# Sydney Girls High School

## 2008 HSC Assessment Task 2

# MATHEMATICS

## Extension 2

**Time Allowed: 90 minutes**

Topic: Complex Numbers

Directions to Candidates:

- There are THREE (3) questions, of equal value.
- All questions must be attempted.
- All necessary working should be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- Write on one side of the paper only.
- Diagrams are NOT to scale.

Question 1: (25 marks)

a) If $z = 3 - 2i$ and $\omega = 1 + 4i$, find in the form $x + iy$:

i. $2z + 3\omega$ — 2

ii. $iz - \omega$ — 2

iii. $\dfrac{\omega}{z}$ — 2

iv. $\overline{z\omega}$ — 3

b) i. Find $\sqrt{-3-4i}$ and express each answer in the form $x + iy$. — 4

ii. Using (i) or otherwise, solve the equation $z^2 - 3z + (3+i) = 0$. — 2

c) i. Express $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ in mod-arg form. — 2

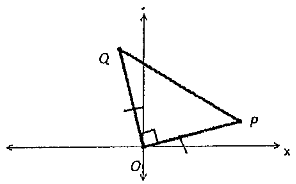
ii. Hence express $z^6$ in the form $x + iy$, where $x$ and $y$ are real numbers. — 2

d) i. If $z = \cos\theta + i\sin\theta$, show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ — 2

ii. Hence or otherwise express $\cos^4\theta$ in terms of multiples of $\theta$. — 4

Question 2: (25 marks)

a) The points $P$ and $Q$ in the complex plane correspond to the complex numbers $z$ and $w$ respectively. The triangle $OPQ$ is isosceles and $\angle POQ$ is a right angle. Show that $z^2 + w^2 = 0$. **2**




b) Sketch the region in the complex plane where the inequalities $|z+1-2i| \le 3$ and $-\frac{\pi}{3} \le \arg z \le \frac{\pi}{4}$ both hold. **3**

c) Sketch the locus of the following. Draw separate diagrams.

i. $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$ **2**

ii. $\text{Re}(z) + \text{Im}(z) = 3$ **2**

d) i. Sketch the locus of $|z-1+2i| = |z+3|$. **2**

ii. Find the locus of $z$. **2**

e) Find the locus of $z$ if $w = \frac{z-2i}{z+2}$ is purely imaginary. **3**

f) i. If $z = x+iy$, sketch on an Argand diagram, the curve defined by the equation $\text{Im}(z-2+i) = 3$. **2**

ii. Find, using your diagram, the minimum value of $|z|$ subject to this condition. **1**

g) i. Prove that $|z|^2 = z\bar{z}$ **1**

ii. Prove that for all complex numbers $z$ and $w$: **2**

$$|z+w|^2 + |z-w|^2 = 2\left(|z|^2 + |w|^2\right)$$

h) The point $A$ on an Argand diagram represents the complex number $1+i$. Find the complex number represented by $B$ if $OBA$ is an equilateral triangle and $B$ is in the second quadrant. **3**

Question 3: (25 marks)

a) i. Find in mod-arg form the five roots of $z^5 = -1$. **2**

ii. Hence factorise $z^5 + 1$ over the real field. **3**

iii. Show that $z^4 - z^3 + z^2 - z + 1 = \left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$. **3**

iv. Show that $\cos\frac{3\pi}{5} + \cos\frac{\pi}{5} = \frac{1}{2}$ and $\cos\frac{3\pi}{5}\cos\frac{\pi}{5} = -\frac{1}{4}$ **2**

v. Deduce that $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ are the roots of the equation $4x^2 - 2x - 1 = 0$. **2**

vi. Find the exact values of $\cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$. **2**

b) $\omega$ is the complex root of $z^6 - 1 = 0$ with the smallest positive argument.

i. Find the two real roots of $z^6 - 1 = 0$. **1**

ii. Prove that $\omega$, $\omega^2$, $\omega^4$ and $\omega^5$ are the roots of $z^4 + z^2 + 1 = 0$. **2**

iii. Find the quadratic equation whose roots are $\alpha = \omega + \omega^5$ and $\beta = \omega^2 + \omega^4$. **3**

c) i. Express $(3+2i)(5+4i)$ and $(3-2i)(5-4i)$ in the form $a+ib$. **2**

ii. Hence find the prime factors of $7^2 + 22^2$. **3**

## 2008 EXTENSION 2 MATHEMATICS – HSC ASSESSMENT TASK 2 SOLUTIONS

Question 1:

a) $z=3-2i$ and $\omega=1+4i$

i. $2z+3\omega=2(3-2i)+3(1+4i)$
$=6-4i+3+12i$
$=9+8i$

ii. $iz-\omega=i(3-2i)-(1+4i)$
$=3i-2i^2-1-4i$
$=3i+2-1-4i$
$=1-i$

iii. $\frac{\omega}{z}=\frac{1+4i}{3-2i}\times\frac{3+2i}{3+2i}$
$=\frac{(1+4i)(3+2i)}{(3-2i)(3+2i)}$
$=\frac{3+14i+8i^2}{9-4i^2}$
$=\frac{3+14i-8}{9+4}$
$=\frac{-5+14i}{13}$

iv. $\overline{z\omega}=\bar{z}\times\bar{\omega}$
$=(3+2i)(1-4i)$
$=3-10i-8i^2$
$=3-10i+8$
$=11-10i$

b) Let $x+iy=\sqrt{-3-4i}$

i. $(x+iy)^2=-3-4i$
$x^2+i2xy+i^2y^2=-3-4i$
$(x^2-y^2)+i2xy=-3-4i$

Equating real and imag. coefficients:
$x^2-y^2=-3$ ---- (1)
$2xy=-4$ ---- (2)

$(x^2+y^2)^2=(x^2-y^2)^2+(2xy)^2$
$(x^2+y^2)^2=(-3)^2+(-4)^2$
$=25$
$x^2+y^2=5$ ---- (3)
$(1)+(3)$:
$2x^2=2$
$x^2=1$
$x=\pm1$

From (2):
When $x=1, y=-2$
When $x=-1, y=2$

$\therefore\sqrt{-3-4i}=\pm(1-2i)$

ii. $z^2-3z+(3+i)=0$
$z=\frac{3\pm\sqrt{9-4(3+i)}}{2}$
$=\frac{3\pm\sqrt{-3-4i}}{2}$
$=\frac{3\pm(1-2i)}{2}$

$z=\frac{3-(1-2i)}{2}$ $\quad z=\frac{3+(1-2i)}{2}$
$=\frac{2+2i}{2}$ $\quad =\frac{4-2i}{2}$
$=1+i$ or $=2-i$

c) $z=\frac{1}{2}+\frac{\sqrt{3}}{2}i$

i. $|z|=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$
$=\sqrt{\frac{1}{4}+\frac{3}{4}}$
$=1$

$\arg z=\tan^{-1}\frac{y}{x}$
$\theta=\tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)$
$=\tan^{-1}\sqrt{3}$
$=\frac{\pi}{3}$

$\therefore z=cis\frac{\pi}{3}$

ii. $z^6=\left(cis\frac{\pi}{3}\right)^6$
$=cis\frac{6\pi}{3}$
$=cis2\pi$
$=\cos2\pi+i\sin2\pi$
$=1$

d)

i. $z=\cos\theta+i\sin\theta$
$z^n+\frac{1}{z^n}=z^n+z^{-n}$
$=(\cos\theta+i\sin\theta)^n+(\cos\theta+i\sin\theta)^{-n}$
$=\cos n\theta+i\sin n\theta+\cos(-n\theta)+i\sin(-n\theta)$
$=\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta$
$=2\cos n\theta$

ii. $z+\frac{1}{z}=2\cos\theta$
$\left(z+\frac{1}{z}\right)^4=(2\cos\theta)^4$
$=16\cos^4\theta$
$\left(z+\frac{1}{z}\right)^4=z^4+4z^2+6+\frac{4}{z^2}+\frac{1}{z^4}$
$16\cos^4\theta=\left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6$
$=2\cos4\theta+8\cos2\theta+6$
$\cos^4\theta=\frac{1}{8}\cos4\theta+\frac{1}{2}\cos2\theta+\frac{3}{8}$

Question 2:

a)

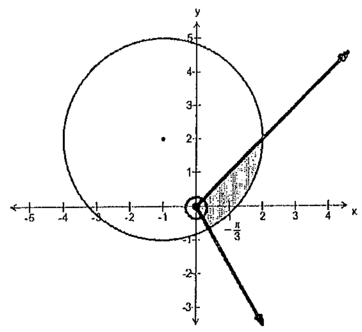
Transformation from $OP$ to $OQ$ is a rotation of $+90^0$.

$w=iz$
$w^2=i^2z^2$
$=-z^2$
$z^2+w^2=0$

b)

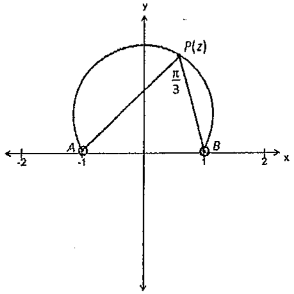
$|z+1-2i|\le3$
$|z-(-1+2i)|\le3$

Circle centre $(-1,2)$ radius 3 units.



c)

i.



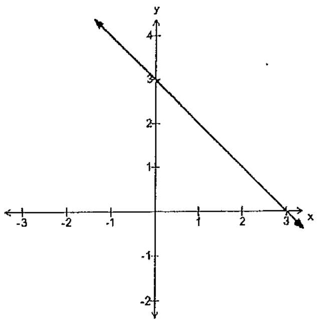

Locus of $z$ is the major arc of a circle excluding $A$ and $B$.

ii. If $z = x + iy$, then:

$\text{Re}(z) = x$

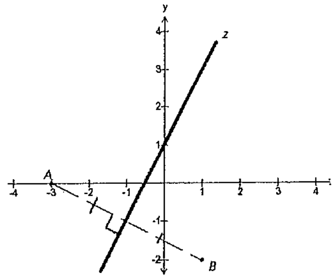
$\text{Im}(z) = y$

$\therefore$ locus of $z$ is $x + y = 3$




d)

i.




ii. locus of $z$ is the perpendicular bisector of $AB$.

Midpoint $AB$:

$$M_{AB} = \left(\frac{-3+1}{2}, \frac{0-2}{2}\right)$$
$$= (-1, -1)$$

Gradient $AB$:

$$m_{AB} = \frac{-2-0}{1+3}$$
$$= -\frac{1}{2}$$

So gradient of locus of $z$ is 2

Equation of locus of $z$:

$$y + 1 = 2(x+1)$$
$$y + 1 = 2x + 2$$
$$2x - y + 1 = 0$$

e) Let $z = x + iy$

*Algebraically:*

$$w = \frac{x + iy - 2i}{x + iy + 2}$$
$$= \frac{x + i(y-2)}{(x+2) + iy} \times \frac{(x+2) - iy}{(x+2) - iy}$$
$$= \frac{x(x+2) - ixy + i(y-2)(x+2) - i^2 y(y-2)}{[(x+2)+iy][(x+2)-iy]}$$

$$w = \frac{x(x+2) - ixy + i(y-2)(x+2) - i^2 y(y-2)}{[(x+2)+iy][(x+2)-iy]}$$
$$= \frac{x^2 + 2x - ixy + i(xy + 2y - 2x - 4) + y^2 - 2y}{(x+2)^2 - i^2 y^2}$$
$$= \frac{x^2 + 2x + y^2 - 2y}{(x+2)^2 + y^2} + i\frac{2y - 2x - 4}{(x+2)^2 + y^2}$$

If $w$ is purely imaginary, then $\text{Re}(z) = 0$.

$$\frac{x^2 + 2x + y^2 - 2y}{(x+2)^2 + y^2} = 0$$
$$x^2 + 2x + y^2 - 2y = 0$$
$$x^2 + 2x + 1 + y^2 - 2y + 1 = 2$$
$$(x+1)^2 + (y-1)^2 = 2$$

Locus of $z$ is a centre, centre $(-1,1)$ and radius $\sqrt{2}$ units, excluding $(0,2)$ and $(-2,0)$.

*Geometrically:*

If $w$ is purely imaginary then $\arg w = \pm\frac{\pi}{2}$.

$$w = \frac{z - 2i}{z + 2}$$
$$\arg w = \arg(z - 2i) - \arg(z + 2)$$
$$\pm\frac{\pi}{2} = \arg(z - 2i) - \arg(z + 2)$$

Locus of $z$ is a circle with (0,2) and (-2,0) as endpoints of the diameter.

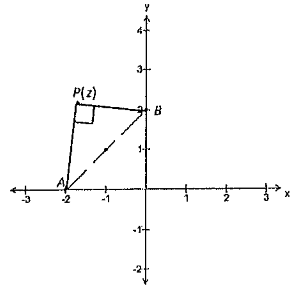
Centre of circle:

$$M_{AB} = \left(\frac{-2+0}{2}, \frac{0+2}{2}\right)$$
$$= (-1, 1)$$

Diameter = $2\sqrt{2}$ (using Pythagoras' Thm:

Radius = $\sqrt{2}$

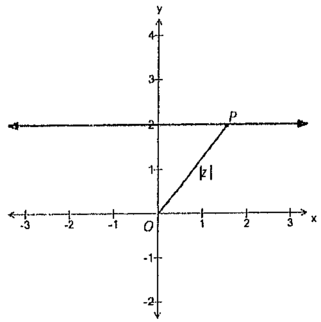
Locus of $z$ is a centre, centre $(-1,1)$ and radius 2 units, excluding $(0,2)$ and $(-2,0)$.




f) If $z = x + iy$, then:

$$\text{Im}(z - 2 + i) = 3$$
$$\text{Im}(x + iy - 2 + i) = 3$$
$$\text{Im}(\{x-2\} + i\{y+1\}) = 3$$
$$y + 1 = 3$$
$$y = 2$$

$$y + 1 = 3$$
$$y = 2$$




If $P$ represents $z$, then $OP = |z|$. When $P$ has coordinates (0,2), minimum $|z| = 2$.

g)

i. $LHS = |z|^2$

$= \left(\sqrt{x^2+y^2}\right)^2$

$= x^2+y^2$

$= x^2 - i^2y^2$

$= (x+iy)(x-iy)$

$= z\bar{z}$

$= RHS$

ii. $LHS = |z+w|^2 + |z-w|^2$

$= (z+w)\left(\overline{z+w}\right) + (z-w)\left(\overline{z-w}\right)$

$= (z+w)\left(\bar{z}+\bar{w}\right) + (z-w)\left(\bar{z}-\bar{w}\right)$

$= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} + z\bar{z} - z\bar{w} - w\bar{z} + w\bar{w}$

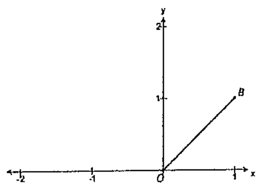
$= 2z\bar{z} + 2w\bar{w}$

$= 2\left(z\bar{z} + w\bar{w}\right)$

$= 2\left(|z|^2 + |w|^2\right)$

$= RHS$

h)




Transformation from $OA$ to $OB$ is a rotation of $+60^\circ$.

$(1+i)cis\frac{\pi}{3}$

$= (1+i)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= (1+i)\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{2} + i\frac{\sqrt{3}}{2} + i\frac{1}{2} + i^2\frac{\sqrt{3}}{2}$

$= \frac{1-\sqrt{3}}{2} + i\frac{\sqrt{3}+1}{2}$

Question 3:

a)

i. $z^5 = -1$

$= cis\pi$

$z = (cis\pi)^{\frac{1}{5}}$

$= cis\left(\frac{\pi+2k\pi}{5}\right)$ where $k = 0,1,2,3,4$

When $k=0$: $z_0 = cis\frac{\pi}{5}$

When $k=1$: $z_1 = cis\frac{3\pi}{5}$

When $k=2$: $z_2 = cis\frac{5\pi}{5}$

$= cis\pi$

$= -1$

When $k=3$: $z_3 = cis\frac{7\pi}{5}$

$= cis\left(-\frac{3\pi}{5}\right)$

$= \overline{z_1}$

When $k=4$: $z_4 = cis\frac{9\pi}{5}$

$= cis\left(-\frac{\pi}{5}\right)$

$= \overline{z_0}$

*QUESTION 3 CONTINUES OVERLEAF...*

ii.

$z^5+1 = (z+1)(z-z_0)\left(z-\overline{z_0}\right)(z-z_1)\left(z-\overline{z_1}\right)$

$= (z+1)\left(z^2 - z\overline{z_0} - zz_0 + z_0\overline{z_0}\right)\left(z^2 - z\overline{z_1} - zz_1 + z_1\overline{z_1}\right)$

$= (z+1)\left(z^2 - \left\{z_0+\overline{z_0}\right\}z + z_0\overline{z_0}\right)\left(z^2 - \left\{z_1+\overline{z_1}\right\}z + z_1\overline{z_1}\right)$

As $z_0+\overline{z_0} = 2\text{Re}(z_0)$ and $z_1+\overline{z_1} = 2\text{Re}(z_1)$

$= 2\cos\frac{\pi}{5}$ $\quad$ $= 2\cos\frac{3\pi}{5}$

$z_0\overline{z_0} = |z_0|^2$ $\quad$ $z_1\overline{z_1} = |z_1|^2$

$= 1$ $\quad$ $= 1$

$z^5+1 = (z+1)\left(z^2 - \left\{z_0+\overline{z_0}\right\}z + z_0\overline{z_0}\right)\left(z^2 - \left\{z_1+\overline{z_1}\right\}z + z_1\overline{z_1}\right)$

$= (z+1)\left(z^2 - 2z\cos\frac{\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{5} + 1\right)$

iii.

$z^4 - z^3 + z^2 - z + 1 = 1 - z + z^2 - z^3 + z^4$

G.P with $a=1$, $r=-z$, $n=5$

$1 - z + z^2 - z^3 + z^4 = \frac{1\left(1-\{-z\}^5\right)}{1-\{-z\}}$

$= \frac{1+z^5}{1+z}$

$= \frac{(z+1)\left(z^2-2z\cos\frac{\pi}{5}+1\right)\left(z^2-2z\cos\frac{3\pi}{5}+1\right)}{1+z}$ from (ii)

$= \left(z^2-2z\cos\frac{\pi}{5}+1\right)\left(z^2-2z\cos\frac{3\pi}{5}+1\right)$

iv.

$$1-z+z^2-z^3+z^4=\left(z^2-2z\cos\frac{\pi}{5}+1\right)\left(z^2-2z\cos\frac{3\pi}{5}+1\right)$$
$$=z^2\left(z^2-2z\cos\frac{3\pi}{5}+1\right)-2z\cos\frac{\pi}{5}\left(z^2-2z\cos\frac{3\pi}{5}+1\right)+1\left(z^2-2z\cos\frac{3\pi}{5}+1\right)$$
$$=z^4-2z^3\cos\frac{3\pi}{5}+z^2-2z^3\cos\frac{\pi}{5}+4z^2\cos\frac{\pi}{5}\cos\frac{3\pi}{5}-2z\cos\frac{\pi}{5}+z^2-2z\cos\frac{3\pi}{5}+1$$
$$=z^4-2z^3\left(\cos\frac{3\pi}{5}+\cos\frac{\pi}{5}\right)+z^2\left(4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}+2\right)-2z\left(\cos\frac{3\pi}{5}+\cos\frac{\pi}{5}\right)+1$$

Equating coefficients of $z^3$:

$$2\left(\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}\right)=1$$
$$\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$$

Equating coefficients of $z^2$:

$$4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}+2=1$$
$$4\cos\frac{\pi}{5}\cos\frac{3\pi}{5}=-1$$
$$\cos\frac{\pi}{5}\cos\frac{3\pi}{5}=-\frac{1}{4}$$

v. Let $\alpha=\cos\frac{\pi}{5}$ and $\beta=\cos\frac{3\pi}{5}$

$\alpha+\beta=\frac{1}{2}$ and $\alpha\beta=-\frac{1}{4}$

$$x^2-(\alpha+\beta)x+\alpha\beta=0$$
$$x^2-\frac{1}{2}x-\frac{1}{4}=0$$
$$4x^2-2x-1=0$$

$\therefore \cos\frac{\pi}{5}$ and $\cos\frac{3\pi}{5}$ are roots of the equation $4x^2-2x-1=0$

vi. $4x^2-2x+1=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$=\frac{2\pm\sqrt{4-4\times4\times(-1)}}{2\times4}$$
$$=\frac{2\pm\sqrt{20}}{8}$$
$$=\frac{2\pm2\sqrt{5}}{8}$$
$$=\frac{1\pm\sqrt{5}}{4}$$

$\cos\frac{\pi}{5}=\frac{1+\sqrt{5}}{4}$ (as $\frac{\pi}{5}$ is in 1st quad.) and $\cos\frac{3\pi}{5}=\frac{1-\sqrt{5}}{4}$ (as $\frac{3\pi}{5}$ is in 2nd quad.)

b)

ii.

$$z^6-1=(z^2)^3-1^3$$
$$=(z^2-1)(z^4+z^2+1)$$

When $z^6-1=0$:

$$(z^2-1)(z^4+z^2+1)=0$$

Real roots are $\pm1$.

iii.

$$z^6-1=0$$
$$z^6=1$$
$$=cis0$$
$$z=cis\frac{k\pi}{3} \text{ where } k=0,1,2,3,4,5$$

When $k=0$: $z_0=cis0$ $=1$

When $k=1$: $z_1=cis\frac{\pi}{3}$

When $k=2$: $z_2=cis\frac{2\pi}{3}$

When $k=3$: $z_3=cis\frac{3\pi}{3}$ $=\cos\pi+i\sin\pi$ $=-1$

When $k=4$: $z_4=cis\frac{4\pi}{3}=cis\left(-\frac{2\pi}{3}\right)=\overline{z_2}$

When $k=5$: $z_5=cis\frac{5\pi}{3}=cis\left(-\frac{\pi}{3}\right)=\overline{z_1}$

$$z^6-1=(z-1)(z+1)(z-z_1)(z-z_2)(z-z_4)(z-z_5)$$
$$(z^2-1)(z^4+z^2+1)=(z^2-1)(z-z_1)(z-z_2)(z-z_4)(z-z_5)$$
$$z^4+z^2+1=(z-z_1)(z-z_2)(z-z_4)(z-z_5)$$

Roots of $z^4+z^2+1=0$ are $z_1$, $z_2$, $z_4$ and $z_5$

Let $\omega=cis\frac{\pi}{3}$ (complex root with the smallest positive argument)

| | | | |
|---|---|---|---|
| $\omega=\left(cis\frac{\pi}{3}\right)$ | $\omega^2=\left(cis\frac{\pi}{3}\right)^2$ | $\omega^4=\left(cis\frac{\pi}{3}\right)^4$ | $\omega^5=\left(cis\frac{\pi}{3}\right)^5$ |
| $=cis\frac{\pi}{3}$ | $=cis\frac{2\pi}{3}$ | $=cis\frac{4\pi}{3}$ | $=cis\frac{5\pi}{3}$ |
| $=z_1$ | $=z_2$ | $=cis\left(-\frac{2\pi}{3}\right)$ | $=cis\left(-\frac{\pi}{3}\right)$ |
| | | $=\overline{z_2}$ | $=\overline{z_1}$ |

So $\omega$, $\omega^2$, $\omega^4$ and $\omega^5$ are the roots of $z^4+z^2+1=0$.

iv.

$$\begin{aligned}\alpha&=\omega+\omega^5\\&=z_1+\overline{z_1}\\&=2\mathrm{Re}(z_1)\\&=2\cos\frac{\pi}{3}\\&=2\times\frac{1}{2}\\&=1\end{aligned}$$

$$\begin{aligned}\beta&=\omega^2+\omega^4\\&=z_2+\overline{z_2}\\&=2\mathrm{Re}(z_2)\\&=2\cos\frac{2\pi}{3}\\&=2\times\left(-\frac{1}{2}\right)\\&=-1\end{aligned}$$

So
$$\begin{aligned}\alpha+\beta&=1+(-1)\\&=0\\\alpha\beta&=1\times(-1)\\&=-1\end{aligned}$$
$$x^2-(\alpha+\beta)x+\alpha\beta=0$$
$$x^2-1=0$$

c)

i.

$$\begin{aligned}(3+2i)(5+4i)&=15+22i+8i^2\\&=15+22i-8\\&=7+22i\end{aligned}$$

$$\begin{aligned}(3-2i)(5-4i)&=15-22i+8i^2\\&=15-22i-8\\&=7-22i\end{aligned}$$

ii.

$$\begin{aligned}(7+22i)(7-22i)&=(3+2i)(5+4i)(3-2i)(5-4i)\\7^2-22^2i^2&=(3+2i)(3-2i)(5+4i)(5-4i)\\7^2+22^2&=(9-4i^2)(25-16i^2)\\&=(9+4)(25+16)\\&=13\times41\end{aligned}$$