Sydney Girls High School

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LABOR OMNIA VINCIT

2003 Assessment Task 2

MATHEMATICS

Extension 2

Year 12

Time allowed - 90 minutes

Topics; Complex Numbers and Polynomials

Instructions

NAME ______________________

- Attempt all questions.
- Questions are NOT of equal value.
- All necessary working should be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- Write on one side of the paper only.
- Questions do not necessarily appear in order of difficulty.

Question One (27 marks)

1. Given $z=1-3i$ find in the form $a+ib$ (where applicable) [7]

   a) $\bar{z}$ b) z^2 c) $z\bar{z}$ d) $\dfrac{1}{z}$ e) $z+iz$

   f) $\arg(z)$ g) $\mathrm{Re}(z)$

2. Given $z_1=a+ib$ and $z_2=c+id$ prove $\overline{z_1}+\overline{z_2}=\overline{z_1+z_2}$ [2]

3. Find the square roots of $5-12i$ [3]

4. Sketch the locus of the following: [6]

   a) $\mathrm{Im}(z)<2 \cap \mathrm{Re}(z)<-1$ b) $|z|<4 \cap \arg(z)\geq\dfrac{\pi}{4}$

   c) $Arg\left(\dfrac{z-2i}{z+2i}\right)=\dfrac{\pi}{2}$

5. a) Find the three cube roots of unity and plot them on an Argand diagram. [3]

   b) If w is the root with the smallest positive argument show the other is w^2 [1]

   c) Show $1+w+w^2=0$ [1]

   d) Evaluate $(4+w+w^2)(2+3w+3w^2)$ [1]

   e) Show $\dfrac{a+bw+cw^2}{b+cw+aw^2}=w$ [1]

   f) Form the quadratic equation with roots $aw+bw^2$ and aw^2+bw [2]

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Question Two (26 marks)

1. a) Write $\left(-\sqrt{3}+i\right)$ in modulus argument form [2]

   b) Hence or otherwise evaluate $\left(-\sqrt{3}+i\right)^6$ [1]

2. If $z=\cos\theta+i\sin\theta$: [7]

   a) Show that $z^n+\frac{1}{z^n}=2\cos n\theta$

   b) Express in $\cos^4\theta$ terms of $\cos n\theta$

   c) Find $\int\cos^4\theta d\theta$

3. a) Sketch the locus of $\arg(z-2)=\frac{2\pi}{3}$ [2]

   b) Hence find z so that $|z|$ is a minimum [2]

4. Find the Cartesian equation of $|z-8|=3|z-2i|$ [3]

5. a) Solve $z^5+1=0$. [4]

   b) Hence factorise z^5+1 over the real field. [3]

   c) Prove that $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$. [2]

Question Three (27 marks)

1. Fully factorise x^4-36 over the complex field [3]

2. Solve the equation $x^4-5x^3-9x^2+81x-108=0$ given that it has a root of multiplicity three. [4]

3. The polynomial equation $2x^3+3x^2-4x-4=0$ has roots α,β and γ. Find the polynomial equation with roots: [4]

   a) $2\alpha, 2\beta$ and 2γ

   b) α^2,β^2 and γ^2

4. Find the equation of a cubic polynomial that is monic, odd and gives a remainder of 4 when divided by $x+1$ [3]

5. Given the polynomial equation $3x^3+5x^2+10x-4=0$ [4]

   a) Show that $\left(-1+\sqrt{3}i\right)$ is a root.

   b) Hence or otherwise find the other roots.

6. Given that $px^4+4qx+r=0$ has a double root α: [5]

   a) Show that $p\alpha^3+q=0$

   b) Hence prove $27q^4=pr^3$

7. Given the polynomial $P(x)=x^3-6x^2+9x+k$ find the values of k for which $P(x)=0$ has exactly one root. [4]

↑
real

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Q1 Soln's. Extn 2 Task 2 2003

1. $z = 1-3i$

a) $\bar{z} = \underline{1+3i}$ #

b) $z^2 = (1-3i)^2$
$= \underline{-8-6i}$ #

c) $z\bar{z} = (1-3i)(1+3i)$
$= \underline{10}$ #

d) $\frac{1}{1-3i} \times \frac{1+3i}{1+3i} = \underline{\frac{1}{10} + \frac{3i}{10}}$ #

e) $z + iz = 1-3i + i(1-3i)$
$= 1-3i+i+3$
$= \underline{4-2i}$ #

f) $\arg z = \underline{\tan^{-1} -\frac{3}{1}}$
$\doteqdot \underline{-71°34'}$ #
$(-1.249\ldots)$ ⑦

g) $Re(z) = \underline{1}$ #

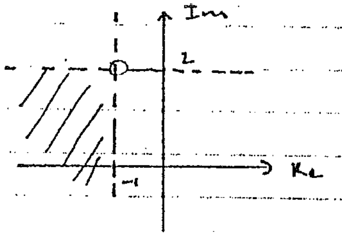
2. $LHS = \overline{a+ib} + \overline{c+id}$
$= a - ib + c - id$
$= (a+c) - i(b+d)$

$RHS = \overline{a+ib+c+id}$
$= \overline{(a+c)+i(b+d)}$
$= (a+c) - i(b+d)$
$= LHS$ # ②

3. let $a+ib = \sqrt{5-12i}$
then $a^2-b^2+2abi = 5-12i$
equate Re, Im
$a^2-b^2 = 5$ ① ①
$2ab = -12$ ②
$(a^2+b^2)^2 = (a^2-b^2)^2+4a^2b^2$
$= 25+144$
$a^2+b^2 = 13$ ③ $a^2+b^2 > 0$
①+③ $2a^2 = 18$
$a = 3$ or $a = -3$
$b = -2$ or $b = 2$
ie $\underline{3-2i}$ # ① , $\underline{-3+2i}$ # ①

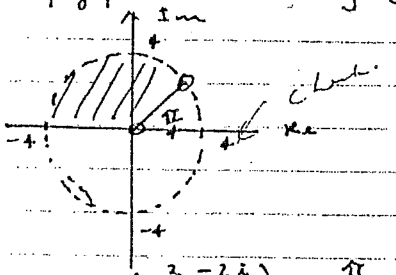
3

4. a) $Im(z) < 2 \wedge Re(z) < -1$
ie $y < 2 \wedge x < -1$



②

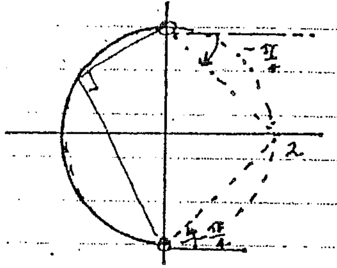
b) $|z| < 4 \wedge Arg(z) \geq \frac{\pi}{4}$



②

c) $Arg\left(\frac{z-2i}{z+2i}\right) = \frac{\pi}{2}$

ie $Arg(z-2i) - Arg(z+2i) = \frac{\pi}{2}$



Check
$-\frac{\pi}{4} - \frac{\pi}{4} = -\frac{\pi}{2}$
$\therefore$ LHS

Q5 part f) cont

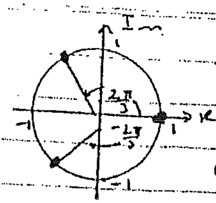
$\alpha\beta = (a^2\omega + b\omega^2)(a\omega^2 + b\omega)$
$= a^2\omega^3 + ab\omega^2 + ab\omega^4 + b^2\omega^3$
$= a^2 + ab\omega^2 + ab\omega + b^2$
$= a^2+b^2 + (ab + ab\omega + ab\omega^2 - ab)$
$= a^2+b^2-ab$
ie $z^2 + (a+b)z + a^2+b^2-ab = 0$ ②

5. a) $z^2 = 1$

$z^3 = \text{cis}\, 0$

make general

$z^3 = \text{cis}\, 2\pi k$, $k = 0, 1, 2$



①

take cube root apply De Moivres Th

$z = \text{cis}\, \frac{2\pi k}{3}$

when $k=0$ $\quad$ $k=1$ $\quad$ $k=2$

$z_0 = \text{cis}\, 0$ $\quad$ $z_1 = \underline{\text{cis}\, \frac{2\pi}{3}}$ $\quad$ $z_2 = \text{cis}\, \frac{4\pi}{3}$ ②

$= \underline{1}$ $\qquad\qquad\qquad\qquad$ $= \text{cis}\, -\frac{2\pi}{3}$

b) $w = \text{cis}\, \frac{2\pi}{3} \Rightarrow w^2 = \text{cis}\, \frac{4\pi}{3}$

$= \text{cis}\, -\frac{2\pi}{3}$

$= z_2$ ①

c) $1 + w + w^2$ are roots of $z^3 - 1 = 0$

Sum of roots $= 1 + w + w^2$

$= \frac{-b}{a}$

$= 0$ ①

d) $(4 + w + w^2)(2 + 3w + 3w^2$

$= (3 + 1 + w + w^2)(3 + 3w + 3w^2 - 1)$

$= -3$

①

e) $\dfrac{a + bw + cw^2}{b + cw + aw^2} \times \dfrac{w}{w}$

$= \dfrac{a + bw + cw^2}{bw + cw^2 + a} \times w$

$= w$ ①

f) eqn of form

$z^2 - (\alpha + \beta)z + \alpha\beta = 0$

$\alpha + \beta = aw + bw^2 + aw^2 + bw$

$= (a + aw + aw^2 - a) + (b + bw + bw^2 - b)$

$= -a - b$

Remainder see end of Q1 part f)

Question Two $\quad$ Extension 2 Task 2 2003

1/ ⓐ $(-\sqrt{3} + i) = 2\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

2

$= \underline{2}\, \text{cis}\, \underline{\frac{5\pi}{6}}$

ⓑ. $(-\sqrt{3} + i)^6 = \left(2\, \text{cis}\, \frac{5\pi}{6}\right)^6$

$= 2^6\, \text{cis}\, \frac{5\pi}{6} \times 6$ By De Moivre's Thm.

1

$= 64\, \text{cis}\, 5\pi$

$= 64\, \text{cis}\, \pi$

$= 64 \times -1 = \underline{-64}$

2/ ⓐ $z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n$

$= \cos n\theta + i\sin n\theta$. (By De Moivre's Thm)

$z^{-n} = (\cos\theta + i\sin\theta)^n$

3

$z^{-n} = (\cos(-n\theta) + i\sin(-n\theta))$ By De Moivre's Thm.)

$\therefore \frac{1}{z^n} = \cos n\theta - i\sin n\theta$ (cos is even, sin is odd)

Now $z^n + \frac{1}{z^n} = \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

2/ b) Now $2\cos\theta = z + \frac{1}{z}$

$\therefore (2\cos\theta)^4 = \left(z + \frac{1}{z}\right)^4$

$16\cos^4\theta = z^4 + 4z^3 \cdot \frac{1}{z} + 6z^2 \cdot \frac{1}{z^2} + 4z \cdot \frac{1}{z^3} + \frac{1}{z^4}$

2

$16\cos^4\theta = z^4 + \frac{1}{z^4} + 4\left(z^2 + \frac{1}{z^2}\right) + 6.$

$16\cos^4\theta. = 2\cos 4\theta + 4(2\cos 2\theta) + 6.$

$\therefore \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

c)

$I = \int \cos^4\theta\, d\theta$

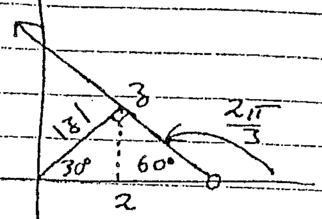
2

$= \int \left(\frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}\right).d\theta.$

$= \frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta + C$

3) a)

2



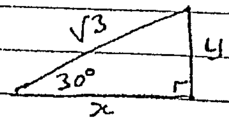
b) $\sin 60 = \frac{|z|}{2}$

$\frac{\sqrt{3}}{2} = \frac{|z|}{2}$

2 $\therefore |z| = \sqrt{3}$

$|z|$ is a minimum when z is perpendicul

OR $\arg z = \frac{\pi}{6}$



$\therefore z = \sqrt{3} \operatorname{cis} \frac{\pi}{6}$

$x = \sqrt{3}\cos 30 = \sqrt{3}.\frac{\sqrt{3}}{2} = \frac{3}{2}$ $y = \sqrt{3}\sin 30 = \sqrt{3}.\frac{1}{2}$.

$z = \frac{3}{2} + \frac{i\sqrt{3}}{2} = \underline{\frac{1}{2}(3+i\sqrt{3})}$

4) $|z-8| = 3|z-2i|$

$\sqrt{(x-8)^2+y^2} = 3\sqrt{x^2+(y-2)^2}$

$x^2-16x+64+y^2 = 9(x^2+y^2-4y+4)$

$x^2-16x+64+y^2 = 9x^2+9y^2-36y+36$

$64 = 8x^2+16x+8y^2-36y+36$

$8 = x^2+2x+y^2-\frac{9}{2}y+\frac{9}{2}$.

$(1+(\frac{9}{4})^2+8 = (x^2+2x+1)+y^2-\frac{9}{2}y+(\frac{9}{4})^2+\frac{9}{2}$.

$8+\frac{81}{16}+1-\frac{9}{2} = (x+1)^2+(y-\frac{9}{4})^2$

3 $\frac{9}{2}+\frac{81}{16} = \frac{72+81}{16} = (x+1)^2+(y-\frac{9}{4})^2$

$\frac{153}{16} = \underline{(x+1)^2+(y-\frac{9}{4})^2}$.

∴ Circle: Centre $(-1, \frac{9}{4})$

Radius $\frac{\sqrt{153}}{4}$

7) (a) Now $z^5+1=0 \Rightarrow z^5=-1$ $\operatorname{cis}\pi = -1$

Let $z = \operatorname{cis}\theta$.

$z^5 = (\operatorname{cis}\theta)^5 = \operatorname{cis}5\theta$ (By De Moivre's Thm)

Hence, $z^5 = \operatorname{cis}(\pi+2k\pi)$

$\therefore \operatorname{cis}5\theta = \operatorname{cis}(\pi+2k\pi)$

$\therefore \theta = \frac{\pi+2k\pi}{5}$ for $k=0,1,2,3,4$

4 $\therefore \theta = \frac{\pi}{5}, \frac{3\pi}{5}, \frac{5\pi}{5}=\pi, \frac{7\pi}{5}=\frac{-3\pi}{5}, \frac{9\pi}{5}=-\frac{\pi}{5}$

∴ Roots of $z^5+1=0$ are $z = \operatorname{cis}\frac{\pi}{5}, \operatorname{cis}\frac{-\pi}{5}, \operatorname{cis}\frac{3\pi}{5}, \operatorname{cis}\frac{-3\pi}{5}$ and $\operatorname{cis}\pi = -1$

Now $z^5+1 = (z+1)(z^4-z^3+z^2-z+1)$.

$\therefore z^4-z^3+z^2-z+1=0$

Has roots $z = \operatorname{cis}\pm\frac{\pi}{5}, \operatorname{cis}\pm\frac{3\pi}{5}, z=-1$

(b) $z^5+1 = (z+1)(z^2-2\cos\frac{\pi}{5}z+1)(z^2-2\cos\frac{3\pi}{5}z+1)$.

as $z = \operatorname{cis}\frac{\pi}{5}$ and $z = \operatorname{cis}\frac{-\pi}{5}$ are conjugate pairs

3 $\therefore (\operatorname{cis}\frac{\pi}{5})(\operatorname{cis}\frac{-\pi}{5}) = 1$

$\operatorname{cis}\frac{\pi}{5}z + \operatorname{cis}\frac{-\pi}{5}z = (\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}z)$

$+(\cos\frac{\pi}{5}-i\sin\frac{\pi}{5})z$

$= 2\cos\frac{\pi}{5}z$.

Same for $\operatorname{cis}\frac{3\pi}{5}, \operatorname{cis}\frac{-3\pi}{5}$

(c) Now $z^5+1 = (z+1)(z^4-z^3+z^2-z+1)$.

$\therefore (z^2-2\cos\frac{\pi}{5}z+1)(z^2-2\cos\frac{3\pi}{5}z+1) = z^4-z^3+z^2-z+1$

Equating coefficients of z

2 $-2\cos\frac{\pi}{5} - 2\cos\frac{3\pi}{5} = -1$

$\therefore \cos\frac{\pi}{5}+\cos\frac{3\pi}{5} = \frac{1}{2}$

No. 1 $x^4-36 = (x^2-6)(x^2+6)$ Soln's Extn 2 Task 2 2003

$= (x-\sqrt{6})(x+\sqrt{6})(x-i\sqrt{6})(x+i\sqrt{6})$

2. $f(x) = x^4-5x^3-9x^2+81x-108$

$f'(x) = 4x^3-15x^2-18x+81$

$f''(x) = 12x^2-30x-18$

$= 6(2x^2-5x-3)$

$= 6(2x+1)(x-3)$

$f(-\frac{1}{2}) \neq 0$, $f(3)=0$ $\therefore f(x) = (x-3)^3(x+4)$

3. $2x^3+3x^2-4x-4=0$ has roots α, β, γ

$\therefore (\frac{x}{2}-\alpha)(\frac{x}{2}-\beta)(x/2-\gamma)=0$ has roots $2\alpha, 2\beta, 2\gamma$

$\Rightarrow 2(\frac{x}{2})^3+3(\frac{x}{2})^2-4(\frac{x}{2})-4=0$

$\therefore \frac{x^3}{4}+\frac{3x^2}{4}-2x-4=0$

or $x^3+3x^2-8x-16=0$

$(\sqrt{x}-\alpha)(\sqrt{x}-\beta)(\sqrt{x}-\gamma)=0$ has roots $\alpha^2, \beta^2, \gamma^2$

$\therefore 2x\sqrt{x}+3x-4\sqrt{x}-4=0$

$\therefore 2\sqrt{x}(x-2) = 4-3x$

$\therefore 4x(x^2-4x+4) = 16-24x+9x^2$

$\therefore 4x^3-16x^2+16x = 9x^2-24x+16$

or $4x^2-25x^2+40x-16=0.$

4. $f(x) = x^3+kx$, $f(-1)=4$

$\therefore 4 = -1-k$ $\Rightarrow k=-5$, $\therefore f(x) = x^3-5x.$

5. a) $3x^3+5x^2+10x-4=0$

If $-1+i\sqrt{3}$ is a root, so is $-1-i\sqrt{3}$

let $S(x) = (x+1+i\sqrt{3})(x+1-i\sqrt{3}) = x^2+2x+4$

$$\begin{array}{r}
3x-1 \\
x^2+2x+4 \,\overline{)\, 3x^3+5x^2+10x-4} \\
\underline{3x^3+6x^2+12x} \\
-x^2-2x-4 \\
\underline{-x^2-2x-4}
\end{array}$$

$\therefore f(x) = (3x-1)(x^2+2x+4)$

has zeros at $1/3$ $-1\pm i\sqrt{3}$

6a) $px^4+4qx+r=0$ has a double root α

let $f(x) = px^4+4qx+r$

$f'(x) = 4px^3+4q$

$f'(\alpha)=0$ $\therefore 4p\alpha^3+4q=0$

$\therefore p\alpha^3+q=0$ $\Rightarrow \alpha^3 = -q/p$

$f(\alpha)=0$, $\therefore p\alpha^4+4q\alpha+r=0$

b) $\therefore \alpha(p\alpha^3+4q) = -r$

$\therefore (-\frac{q}{p})^{1/3}\{-q+4q\} = -r$

$-\frac{q}{p}(27q^3) = -r^3$

$\therefore 27q^4 = pr^3$

7. $P(x) = x^3-6x^2+9x+k$

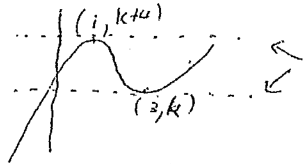
$P'(x) = 3x^2-12x+9$

$= 3(x^2-4x+3)$

$= 3(x-1)(x-3)$

$P(1) = 1-6+9+k$ $P(3) = 27-54+27+k$

$= k+4$ $= 4$



Now $k+4<0$ or $k>0$

$\therefore k<-4$ or $k>0$ gives one real root.