SYDNEY GIRLS HIGH SCHOOL

2005 HSC Assessment Task 2

March 9, 2005

MATHEMATICS Extension 2

Year 12

Time allowed: 90 minutes

Topic: Complex Numbers

DIRECTIONS TO CANDIDATES:

- Attempt all questions
- Questions are not of equal value
- There are 17 questions
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work
- Board approved calculators may be used
- Write on one side of the paper only

Question 1: Given z = 4 + 3i,

a) find (i) iz

(ii) $\overline{z}$

(iii) z + iz

(iv) $z + \overline{z}$

b) plot the points represented by z, iz, $\overline{z}$. z + iz, $z + \overline{z}$ on an Argand Diagram

[6]

Question 2: Given $z = 1 + i$ and $w = 1 - i\sqrt{3}$, write z and w in mod-arg form and hence or otherwise evaluate $\left(\frac{w}{z}\right)^{10}$ in the form a + ib

[6]

Question 3: Sketch the region on the Argand Diagram where:

$$\{3 \leq |z| \leq 5\} \cap \{\frac{\pi}{3} \leq \arg z \leq \frac{2\pi}{3}\}$$

[4]

Question 4: a) Sketch the locus of z given $\arg(z-3) = \frac{2\pi}{3}$ and

b) Find the minimum values of i) $|z|$

ii) $|z+2|$

[5]

Question 5: Solve for z in the form a + ib if:

$$z^2 - z + 4 = i(z + 7)$$

[6]

Question 6: a) Given $|z-2-2i|=|z+4+6i|$,

i) Find the equation of the locus of z

ii) Describe this locus

b) Given $|z-2-2i|=\sqrt{2}|z+4+6i|$

i) Find the equation of the locus of z

ii) Describe this locus

[6]

Question 7: Assuming the result for De Moivre's Theorem for both positive and negative integers and given $z = \cos\theta + i\sin\theta$

a) Show $z^n + z^{-n} = 2\cos n\theta$

b) Express $\cos^5\theta$ in the form $p\cos 5\theta + q\cos 3\theta + r\cos\theta$

c) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \cos^5\theta . d\theta$

[6]

Question 8: Given O is the origin and A is represented by the complex number $\sqrt{3} + i$,

a) Find C if OA is rotated through $\left(-\frac{\pi}{3}\right)$ and doubled in length to form C.

b) Find the point B which forms the parallelogram OABC

[4]

Question 9: Given w is a complex cube root of unity

a) Show that $1 + w + w^2 = 0$

b) Simplify $(1+2w+2w^2)^3$

c) Find the quadratic equation which has the roots $(1 + 3w + w^2)$ and $(1 + w + 3w^2)$

[5]

Question 10: a) Given $|z + 1 - 2i| = 1$, sketch the locus of z

b) Hence or otherwise find

i) the maximum value of arg z to the nearest degree

ii) the maximum value of $|z|$ [6]

Question 11: By expanding $(\cos\theta + i\sin\theta)^4$ two different ways and by finding suitable expressions for $\cos 4\theta$ and $\sin 4\theta$, find $\tan 4\theta$ in terms of $\tan\theta$

[6]

Question 12: Find and describe the locus of z given $w = \dfrac{z-2}{z-2i}$ and w is a purely imaginary number.

[6]

Question 13: If A is represented by the complex number z = 1 + 2i and B is represented by the complex number w = 4 + 4i, find the complex number D which represents the rotation of B about A by $+90^o$.
Hence or otherwise find the point C which completes the square ABCD.

[4]

Question 14: Find the four complex roots of $-8 - 8\sqrt{3}\,i$, giving your answers in the form a + ib

[6]

Question 15: Determine the locus of w, if $|z| = 1$ and $w = \dfrac{z - 1 - i}{z + 1 + i}$.

Describe this locus

[6]

Question 16: Find and sketch the locus of z given $\arg z^2 + \arg(-iz) = 0$

[5]

Question 17: Consider the equation $z^5 = 1$

a) Solve over the complex field
b) Factorise $z^5 - 1$ over the complex field
c) Factorise $z^5 - 1$ over the real field
d) Factorise $z^5 - 1$ over the rational field
e) Show $\cos\dfrac{2\pi}{5} + \cos\dfrac{4\pi}{5} = -\dfrac{1}{2}$
f) Find the exact value of $\cos\dfrac{2\pi}{5}$
g) Hence find the exact value of the area of the pentagon formed by the 5 complex roots of unity. [Do not attempt to simplify the answer]

[13]

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Question One (6 marks)

(a) $z = 4+3i$

(i) $iz = (4+3i)i = 4i + 3i^2 = -3+4i$

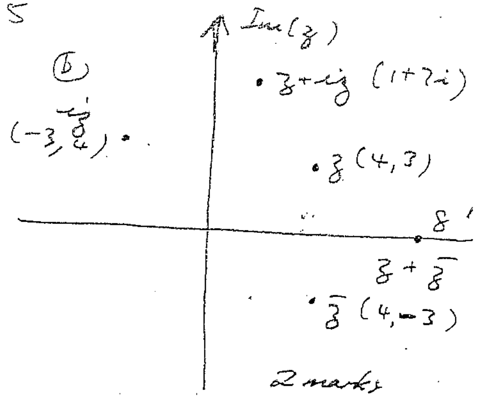
(ii) $\bar{z} = 4-3i$

(iii) $z + iz = 4+3i - 3+4i = 1+7i$

(iv) $z + \bar{z} = 4+3i + 4-3i = 8$.

4 marks

(b)



2 marks

Question Two

$z = 1+i$

$z = \sqrt{2}\,\text{cis}\frac{\pi}{4}$ 1 mark

$w = 1-\sqrt{3}i$ 1 mark

$w = 2\,\text{cis}\left(-\frac{\pi}{3}\right)$.

$z^{10} = \left(\sqrt{2}\,\text{cis}\frac{\pi}{4}\right)^{10}$
$= 2^5 \text{cis}\frac{10\pi}{4}$
$= 32\,\text{cis}\frac{\pi}{2}$
$= 32i$

$w^{10} = 2^{10}\text{cis}\,\frac{-10\pi}{3}$
$= 2^{10}\text{cis}\frac{2\pi}{3}$

$\left(\frac{w}{z}\right)^{10} = \dfrac{2^{10}\text{cis}\frac{2\pi}{3}}{2^5\text{cis}\frac{\pi}{2}}$

$= 2^5\text{cis}\frac{\pi}{6}$

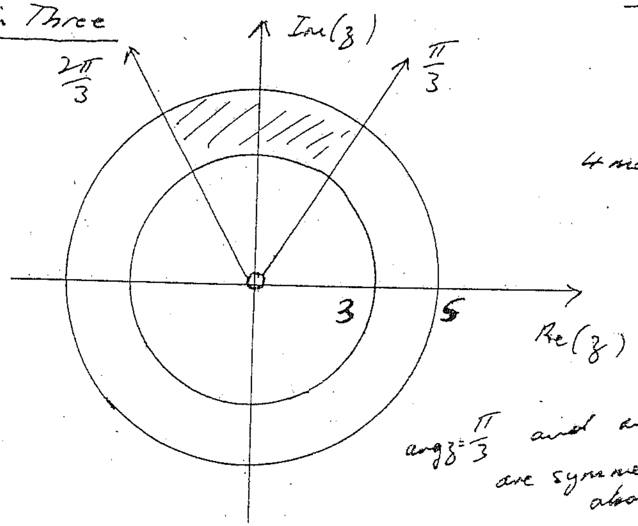
$= 32\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= 16\sqrt{3} + 16i$

in form $a+ib$

4 marks

Question Three

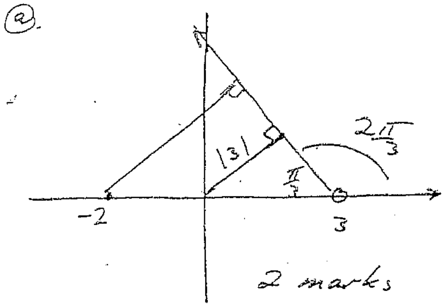


4 marks.

$\arg z = \frac{\pi}{3}$ and $\arg z = \frac{2\pi}{3}$ are symmetrical about $Im(z)$ axis

Question Four (5 marks)

(a)



2 marks

b) (i) $\min|z|$. $\frac{|z|}{3} = \sin\frac{\pi}{3}$

2 marks

$\therefore |z| = 3 \times \frac{\sqrt{3}}{2}$

$|z| = \frac{3\sqrt{3}}{2}$.

(ii) $\min|z+2|$ $\frac{|z+2|}{5} = \sin\frac{\pi}{3}$

$\therefore |z+2| = \frac{5\sqrt{3}}{2}$

1 mark

Question Five (6 marks)

$z^2 - z + 4 = i(z+7)$

$z^2 - z + 4 = iz + 7i$

$z^2 - z(1+i) + (4-7i) = 0$ 1

$z = \dfrac{(1+i) \pm \sqrt{(1+i)^2 - 4(4-7i)}}{2}$

$z = \dfrac{(1+i) \pm \sqrt{-16+30i}}{2}$ 1

Let $x+iy = \sqrt{-16+30i}$

$x^2 - y^2 + 2xyi = -16+30i$. 1

$2xy = 30$ $xy = 15$

$y = 5,\ x = 3$

$\therefore x+iy = 3+5i$

Now $z = \dfrac{(1+i) \pm (3+5i)}{2}$

$z = 2+3i,\ z = -1-2i$ 2

Question Six

(a)(i) $|z-2-2i| = |z+4+6i|$

$\sqrt{(x-2)^2+(y-2)^2} = \sqrt{(x+4)^2+(y+6)^2}$

$x^2 - 4x + 4 + y^2 - 4y + 4$

$= x^2 + 8x + 16 + y^2 + 12y + 36$.

$\Rightarrow 0 = 12x + 16y + 44$

$3x + 4y + 11 = 0$

2 2

(ii) Locus is a straight line which is the perpendicular bisector of the interval joining $(2,2)$ and $(-4,-6)$. 1

(b) $|z-2-2i| = \sqrt{2}\,|z+4+6i|$

$\sqrt{(x-2)^2+(y-2)^2} = \sqrt{2}\sqrt{(x+4)^2+(y+$

$x^2 - 4x + 4 + y^2 - 4y + 4$

$= 2\{x^2 + 8x + 16 + y^2 + 12y + 36\}$

$0 = x^2 + y^2 + 20x + 28y + 96$

$(x+10)^2 + (y+14)^2 = 200$.

(ii) Locus is a circle centre $(-10,-14)$ radius $10\sqrt{2}$.

Question Seven /6

a) $z = \cos\theta + i\sin\theta$

then $z^n = (\cos\theta + i\sin\theta)^n$

$= \cos(n\theta) + i\sin(n\theta)$ ①

and $z^{-n} = (\cos\theta + i\sin\theta)^{-n}$

$= \cos(-n\theta) + i\sin(-n\theta)$

$\frac{1}{z^n} = \cos(n\theta) - i\sin(n\theta)$ ②

①+② $z^n + \frac{1}{z^n} = 2\cos(n\theta)$ 2/

b) put $n=1$ then $z + \frac{1}{z} = 2\cos\theta$

ie $(2\cos\theta)^5 = \left(z + \frac{1}{z}\right)^5$

$= z^5 + 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) + 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) + \frac{1}{z^5}$

$= z^5 + 5z^3 + 10z + \frac{10}{z} + \frac{5}{z^3} + \frac{1}{z^5}$

$= \left(z^5 + \frac{1}{z^5}\right) + 5\left(z^3 + \frac{1}{z^3}\right) + 10\left(z + \frac{1}{z}\right)$

$32\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$

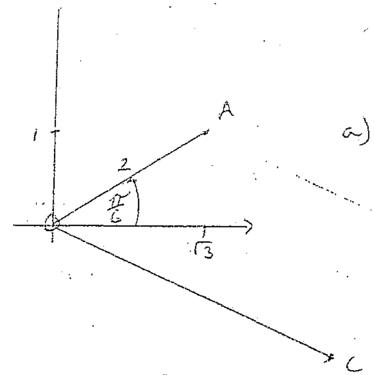
$\cos^5\theta = \frac{1}{16}\cos 5\theta + \frac{5}{16}\cos 3\theta + \frac{10}{16}\cos\theta$ 2/

c) $\int \cos^5\theta \, d\theta = \frac{1}{80}\sin 5\theta + \frac{5}{48}\sin 3\theta + \frac{5}{8}\sin$ ①

$\int_0^{\frac{\pi}{2}} \cos^5\theta \, d\theta = \frac{1}{80} - \frac{5}{48} + \frac{5}{8}$

$= \frac{8}{15}$ ① 2/

Question Eight /4



$\sqrt{3} + i = 2\,\text{cis}\,\frac{\pi}{6}$

a) $\underline{C}$, $2\,\text{cis}\,\frac{\pi}{6} \times 2\,\text{cis}\left(-\frac{\pi}{3}\right)$

$= 4\,\text{cis}\left(-\frac{\pi}{6}\right)$

$= 4\left[\frac{\sqrt{3}}{2} - \frac{i}{2}\right]$

$= 2(\sqrt{3} - i)$ 2/

b) B at $\sqrt{3} + i + 2\sqrt{3} - 2i = 3\sqrt{3} - i$ 2/

Question Nine /5

a) $1 + \omega + \omega^2 = 0$ geometric $a = 1$, $\omega = r$, $\omega^3 = 1$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$= \frac{1(\omega^3 - 1)}{\omega - 1}$

$= 0$ since $\omega^3 = 1$ or or or 1/

b) $(1 + 2\omega + 2\omega^2)^3 = [(1 + \omega + \omega^2) + \omega + \omega^2]^3$

$= [(1 + \omega + \omega^2) - 1]^3$

$= -1$ 2/

c) $\alpha = 1 + 3\omega + \omega^2$, $\beta = (1 + \omega + 3\omega^2)$

$= [(1 + \omega + \omega^2) + 2\omega]$ $= [(1 + \omega + \omega^2) + 2\omega^2]$

$= 2\omega$ $= 2\omega^2$

$\alpha + \beta = [(2\omega + 2\omega^2 + 2) - 2]$, $\alpha\beta = 4\omega^3$

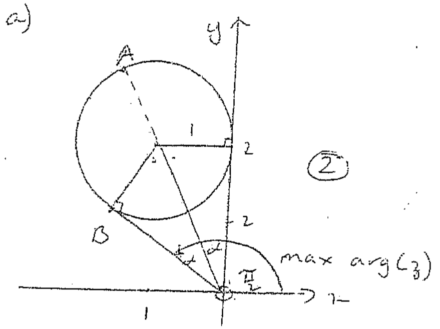
$= -2$ ① $= 4$ ①

ie $x^2 - (-2)x + 4 = 0$

$x^2 + 2x + 4 = 0$ 2/

Question Ten. /6

a)



$②$

b) i) Max $\arg(z) = \frac{\pi}{2} + 2\alpha$

$\alpha = \tan^{-1}\frac{1}{2}$ (at B)

$= 26°34'$

$\therefore$ Max $\arg(z) = 90° + 2 \times 26°34'$

$= 143°8'$

(2.498 radians) $②$

ii) Max $|z|$ at A $= \sqrt{2^2+1^2} + 1$

$= \sqrt{5} + 1$ $②$

Question Eleven /6

$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ (De Moivre's Th)

1
1 2 1
1 3 3 1
1 4 6 4 1

also $(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta$
$+ 4\cos\theta\, i^3\sin^3\theta + i^4\sin^4\theta$

$②$

ie $\cos 4\theta + i\sin 4\theta = \cos^4\theta + 4\cos^3\theta\, i\sin\theta - 6\cos^2\theta\sin^2\theta - 4\cos\theta\, i\sin^3\theta + \sin^4\theta$

Equate Real

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ $①$

Equate Imaginary

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ $①$

Now $\tan 4\theta = \dfrac{\sin 4\theta}{\cos 4\theta}$

$①$

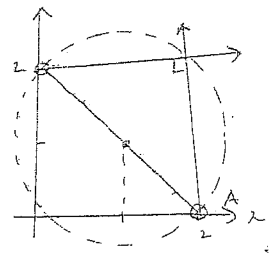
$= \dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$ $\begin{matrix}\div\cos^4\theta\\ \div\cos^4\theta\end{matrix}$

$= \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ $①$

Question Twelve. /6

Geometrically

$②$



$w = \dfrac{z-2}{z-2i}$

$\arg w = \arg(z-2) - \arg(z-2$

If purely imaginary

$\arg w = \pm\frac{\pi}{2}$

$①$

ie $\arg(z-2) - \arg(z-2i) = \pm\frac{\pi}{2}$

Locus is a circle diameter AB, centre C(1,

Now radius $= \sqrt{1^2+1^2}$ — $①$

$= \sqrt{2}$

ie $(x-1)^2 + (y-1)^2 = 2$ $②$

ie circle centre $(1, 1)$ radius $\sqrt{2}$ $①$

exclude $(0, 2)$ $(2, 0)$ $①$

Algebraically

$②$

$z - 2i = x + iy - 2i$

$= x + i(y-2)$

$\therefore \overline{z - 2i} = x - i(y-2)$

$= \bar{z} + 2i$

$w = \dfrac{z-2}{z-2i} \times \dfrac{\bar{z}+2i}{\bar{z}+2i}$

$= \dfrac{z\bar{z} + 2iz - 2\bar{z} - 4i}{(\quad)(\quad)}$

$= \dfrac{x^2+y^2 + 2i(x+iy) - 2(x-iy) - 4}{(\quad)(\quad)}$

$= \dfrac{x^2+y^2 + 2ix - 2y - 2x + 2iy + 8}{(\quad)(\quad)}$

Now if purely imaginary $Re(z) = 0$

$x^2 + y^2 - 2y - 2x = 0$

$x^2 - 2x + y^2 - 2y = 0$

$x^2 - 2x + 1 + y^2 - 2y + 1 = 2$

$(x-1)^2 + (y-1)^2 = (\sqrt{2})^2$ $②$

circle centre $(1, 1)$ radius $\sqrt{2}$ $①$

exclude $(0, 2)$, $(2, 0)$ $①$

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11. $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ (de m's thm)

$= \cos^4\theta + 4\cos^3\theta\, i\sin\theta - 6\cos^2\theta\sin^2\theta - 4\cos\theta\, i\sin^3\theta + \sin^4\theta$

Equating Re & Im.

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

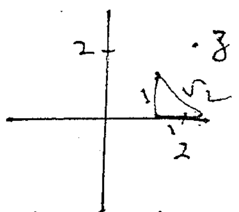
$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$\therefore \tan 4\theta = \dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$

$= \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$.

12. $w = \dfrac{z-2}{z-2i}$

if w is imaginary
$\arg w = \pm\pi/2$

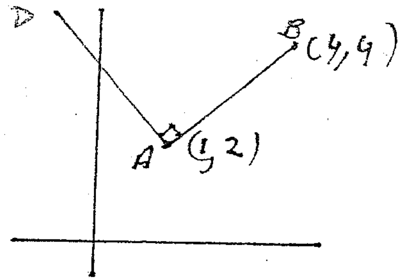


z lies on a circle, diameter $(2,0)$ to $(0,2)$ from diagram

$\therefore$ Locus is $(x-1)^2 + (y-1)^2 = \sqrt{2}$

excluding the points $(0,2)$ & $(2,0)$

13.



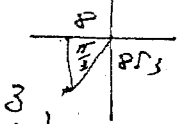
$A \to B$ is $A + 3 + 2i$

if $3+2i$ is rotated $\pi/2$, new point is $(3+2i)i = -2+3i$

$\therefore A \to D$ is $A - 2 + 3i = -1 + 5i$

C is $B - 2 + 3i = 4 + 4i - (+2 - 3i)$
$= 2 + 7i$

Q14. $z^4 = -8 - 8\sqrt{3}\,i$



let $z = r(\cos\theta + i\sin\theta)$

$\therefore z^4 = r^4(\cos 4\theta + i\sin 4\theta) = -8 - 8\sqrt{3}\,i$
(de m's thm)

$|z|^4 = r^4 = 16$ $\quad \arg z^4 = \dfrac{4\pi}{3}, \dfrac{10\pi}{3}, \dfrac{16\pi}{3}, \dfrac{22\pi}{3}$

$r = 2$

$\therefore \arg z = \dfrac{\pi}{3}, \dfrac{5\pi}{6}, \dfrac{4\pi}{3}, \dfrac{11\pi}{6}$

$z_1 = (\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 1 + \sqrt{3}\,i$

$z_2 = (\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}) = -\sqrt{3} + i$

$z_3 = (\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}) = -1 - \sqrt{3}\,i$

$z_4 = (\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}) = \sqrt{3} - i$

Q15. $w = \dfrac{z-1-i}{z+1+i}$

$\therefore wz + w + iw = z - 1 - i$

$wz - z = -w - 1 - iw - i$

$z(1-w) = w + 1 + i(w+1)$

$z = \dfrac{w+1+i(w+1)}{1-w}$

$|z| = 1$, $\therefore \dfrac{|w+1+i(w+1)|}{|1-w|} = 1$

$\therefore |(w+1) + i(w+1)| = |1-w|$

let $w = x + iy$ & square both s

$\therefore (x+1)^2 + y^2 + (x+1)^2 + y^2 = (x-1)^2 + y^2$

$x^2 + 2x + 1 + y^2 + x^2 + 2x + x = x^2 - 2x$

$x^2 + 6x + 1 + y^2 = 0$

$(x+3)^2 + y^2 = 8$

$\therefore$ Locus of w is a circ
centre $(-3, 0)$ & radius $2\sqrt{2}$
or $|w+3| = 2\sqrt{2}$.

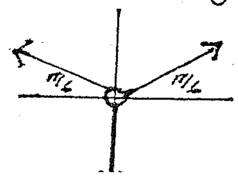
16. $\arg z^2 + \arg(-iz) = 0$

$\therefore 2\arg z + \arg(-i) + \arg z =$

$\therefore 3\arg z - \pi/2 = 0$

$\therefore 3\arg z = \pi/2, 5\pi$

$\arg z = \dfrac{\pi}{6}, \dfrac{5\pi}{6},$



$z^5 = 1$

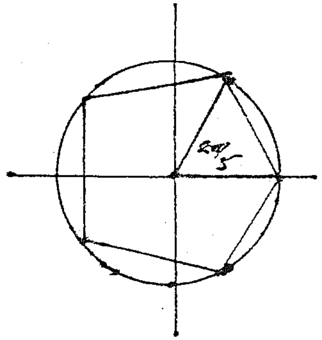
Let $z = \cos\theta + i\sin\theta$

$\therefore z^5 = (\cos\theta + i\sin\theta)^5$

$= \cos 5\theta + i\sin 5\theta$ (de M's Thm)

$|z^5| = 1$, $\arg z^5 = 0, 2\pi, 4\pi, 6\pi, 8\pi$

$\therefore \theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$



$\therefore z_1 = \text{cis}\, 0$

$z_2 = \text{cis}\, \frac{2\pi}{5}$

$z_3 = \text{cis}\, \frac{4\pi}{5}$

$z_4 = \text{cis}\, \frac{6\pi}{5}$

$z_5 = \text{cis}\, \frac{8\pi}{5}$

$z^5 - 1 = (z-1)(z - \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5})(z - \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5})(z - \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5})(z - \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5})$

$= (z-1)(z^2 - 2z\cos\frac{2\pi}{5} + 1)(z^2 - 2z\cos\frac{4\pi}{5} + 1)$

$= (z-1)(z^4 + z^3 + z^2 + z + 1)$

$z^4 + z^3 + z^2 + z + 1 = (z^2 - 2z\cos\frac{2\pi}{5} + 1)(z^2 - 2z\cos\frac{4\pi}{5} + 1)$

$= z^4 - (2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5})z^3 + (2 + 4\cos\frac{2\pi}{5}\cos\frac{4\pi}{5})z^2 - \ldots$

Equate z^3: $1 = -(2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5})$

$\therefore 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = -1$

$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$

Let $\frac{2\pi}{5} = \theta$, $\therefore \frac{4\pi}{5} = 2\theta$

$\Rightarrow$

$\cos 2\theta + \cos\theta = -\frac{1}{2}$

$\therefore 2\cos^2\theta - 1 + \cos\theta = -\frac{1}{2}$

$\therefore 2\cos^2\theta + \cos\theta - \frac{1}{2} = 0$

$4\cos^2\theta + 2\cos\theta - 1 = 0$

$\therefore \cos\theta = \frac{-2 \pm \sqrt{4+16}}{8}$

$= \frac{-2 \pm 2\sqrt{5}}{8}$

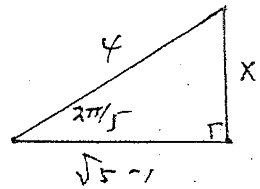
$= \frac{-1 \pm \sqrt{5}}{4}$

But $\cos\frac{2\pi}{5} > 0$

$\therefore \cos\frac{2\pi}{5} = \frac{-1+\sqrt{5}}{4}$

9) $A = 5 \times \frac{1}{2} ab \sin\frac{2\pi}{5}$

$= \frac{5}{2} \times 1 \times 1 \sin\frac{2\pi}{5}$



$X^2 = 16 - (\sqrt{5}-1)^2$

$= 16 - (5 + 1 - 2\sqrt{5})$

$= 10 + 2\sqrt{5}$

$\therefore X = \sqrt{10 + 2\sqrt{5}}$

$\sin X = \frac{\sqrt{10+2\sqrt{5}}}{4}$

& Area is $\frac{5}{8}\sqrt{10+2\sqrt{5}}$