Sydney Girls High School

LABOR OMNIA VINCIT

2008 Assessment Task 3

MATHEMATICS

Extension Two

Year 12

Time allowed - 90 minutes (plus 5 minutes reading time)

Topics: Polynomials and Integration

Instructions

- Attempt all three questions.
- All necessary working should be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- Write on one side of the paper only.
- A table of standard integrals is supplied.

Question One (30 marks) — **Marks**

a) Find $\int \frac{3}{\sqrt{9-x^2}}dx$ — 2

b) Find $\int \frac{dx}{\sqrt{x}(x+1)}$ using the substitution $u = x^{\frac{1}{2}}$ — 3

c) Find $\int xe^{3x}dx$ — 5

d) i) Show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ — 3

ii) Hence or otherwise find $\int_0^2 x(2-x)^6\, dx$ — 3

e) i) Find A, B and C if $\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$ — 3

ii) Hence find $\int \frac{dx}{x(x^2+1)}$ — 2

f) Find $\int \frac{dx}{\sqrt{x^2-x-1}}$ — 4

g) Use the substitution $t = \tan\frac{\theta}{2}$ to find $\int \frac{d\theta}{1+\sin\theta+\cos\theta}$ — 5

Question Two (30 marks) **Marks**

a) Solve the equation $x^3 - 20x^2 + 125x - 250 = 0$ given that it has a double root and the single root is twice the double root. 2

b) Factorise $x^2 + 6x + 18$ over the complex field. 3

c) The polynomial equation $P(x) = x^3 - 6x^2 + 5x - 3$ has roots α, β and γ. Find the value of:

i) $\alpha^2 + \beta^2 + \gamma^2$ 3

ii) $\alpha^3 + \beta^3 + \gamma^3$ 3

iii) $\sum \alpha^2 \beta$ 3

d) Solve the equation $8x^4 + 44x^3 + 54x^2 + 25x + 4 = 0$ given that it has a root of multiplicity three 4

e) The polynomial equation $2x^3 - x^2 + 2x + 6 = 0$ has roots α, β and γ. Find the equation with roots:

i) $(\alpha - 1), (\beta - 1), (\gamma - 1)$ 3

ii) $\alpha^2, \beta^2, \gamma^2$ 3

f) Given that $(x + i)$ is a factor of the polynomial equation $x^4 + 2x^3 - 2x^2 + 2x - 3 = 0$ find all the roots of the equation 4

g) Find a relationship between p, q, r and s if $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ have a common root 2

Question Three (30 marks) **Marks**

a) Evaluate $\int_{-3}^{3} (\cos^5 x \sin^5 x)\,dx$ 2

b) Evaluate $\int_{-3}^{3} \sqrt{9 - x^2}\,dx$ 2

c) Find $\int \frac{\sin^3 x}{\cos^5 x}\,dx$ 3

d) i) Use a substitution to find $\int (\sin^n ax \cos ax)\,dx$ where a is a constant and n an integer. 2

ii) Hence or otherwise show that $\int \cot(ax)\,dx = \frac{1}{a}\log_e |\sin ax| + C$ 2

e) A polynomial P(x) is even. It has a single root at $x = 1$, a double root at $x = 2$ and passes through the point with co ordinates (3, 150). Find the equation of P(x) 3

f) Find $\int \frac{\sin\theta}{\cos^2\theta + \cos\theta - 2}\,d\theta$ 6

g) Given the polynomial $P(x) = x^3 - 9x^2 + 24x + k$, find the values of k for which the polynomial has exactly one real root. 5

h) Given $I_n = \int_0^{\frac{\pi}{2}} e^x \cos^n x\,dx$, find a reduction formula for I_n in terms of I_{n-2} 5

$\alpha\beta\chi\delta\varepsilon\phi\varphi\gamma\eta\iota\kappa\lambda\mu\nu o\pi\varpi\theta\vartheta\rho\sigma\varsigma\tau\upsilon\omega\xi\psi$

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1(a) $3\sin^{-1}\frac{x}{3}+c$

b) $\frac{du}{dx}=\frac{1}{2}x^{-\frac{1}{2}}$

$2\,du\,x^{\frac{1}{2}}=dx$

$\int\frac{2x^{\frac{1}{2}}\,du}{\sqrt{x}\,(u^2+1)}$

$=2\int\frac{du}{u^2+1}$

$=2\tan^{-1}u+c$

$=2\tan^{-1}\sqrt{x}+c$

c) let $u=x$ $\quad v'=e^{3x}$

$u'=1$ $\quad v=\frac{1}{3}e^{3x}$

$\frac{xe^{3x}}{3}-\int\frac{e^{3x}}{3}\,dx$

$=\frac{xe^{3x}}{3}-\frac{e^{3x}}{9}+c$

d) i) let $u=a-x$

$\frac{du}{dx}=-1$ $\quad$ when $x=a$, $u=a-a=0$

$-du=dx$ $\quad$ $x=0$, $u=a-0=a$

$\int_0^a f(x)\,dx$

$=-\int_a^0 f(a-u)\,du$

$=\int_0^a f(a-u)\,du$

$=\int_0^a f(a-x)\,dx$

ii) $\int_0^2 (2-x)x^6\,dx$

$=\int_0^2(2x^6-x^7)\,dx$

$=\left[\frac{2x^7}{7}-\frac{x^8}{8}\right]_0^2$

$=\left(\frac{2\times2^7}{7}-\frac{2^8}{8}\right)-0$

$=4\frac{4}{7}$

(e) i) $1=A(x^2+1)+(Bx+C)x$

when $x=0$ $\quad 1=A$

" $x=1$ $\quad 1=2+B+C$

$-1=B+C$

" $x=2$ $\quad 1=5+4B+2C$

$-4=4B+2C$

$-2=2B$

$B=-1$ $\quad C=0$

ii) $\int\frac{dx}{x}+\int\frac{-x}{x^2+1}\,dx$

$=\log x-\frac{1}{2}\log(x^2+1)+c$

f) $\int\frac{dx}{\sqrt{(x-\frac{1}{2})^2-\frac{5}{4}}}=\int\frac{dx}{(x-\frac{1}{2})^2-(\frac{\sqrt{5}}{2})^2}$ $\quad$ let $u=x-\frac{1}{2}$, $\frac{du}{dx}=1$, $du=dx$

$=\int\frac{dx}{u^2-(\frac{\sqrt{5}}{2})^2}$

$=\log\left(u+\sqrt{u^2-(\frac{\sqrt{5}}{2})^2}\right)+c$

$=\log\left(x-\frac{1}{2}+\sqrt{x^2-x+\frac{1}{4}-\frac{5}{4}}\right)+c$

$=\log\left(x-\frac{1}{2}+\sqrt{x^2-x-1}\right)+c$

g) $\frac{dt}{d\theta}=\frac{1}{2}\sec^2\frac{\theta}{2}$

$=\frac{1+t^2}{2}$

$\frac{2dt}{1+t^2}=d\theta$

$\int\frac{2dt}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\times\frac{1}{1+t^2}$

$=\int\frac{2dt}{1+t^2+2t+1-t^2}$

$=\int\frac{2dt}{2+2t}$

$=\log(2+2t)+c$

$=\log(1+t)+\log 2+c$

$=\log(1+\tan\frac{\theta}{2})+c_2$

Question Two:

a) $x^3 - 20x^2 + 125x - 250 = 0$

Let the roots be $\alpha, \alpha, 2\alpha$

Sum of roots 1 at a time: $4\alpha = 20$

$\alpha = 5$

$\therefore$ Roots are 5, 5, 10 (2)

b) $x^2 + 6x + 18 = x^2 + 6x + 9 + 9$

$= (x+3)^2 - 9i^2$

$= (x+3+3i)(x+3-3i)$ (3)

c) $P(x) = x^3 - 6x^2 + 5x - 3$

When $P(x) = 0$: $\alpha + \beta + \gamma = 6$

$\alpha\beta + \alpha\gamma + \beta\gamma = 5$

$\alpha\beta\gamma = 3$

i) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 6^2 - 2(5)$

$= 26$ (3)

ii) $\alpha^3 - 6\alpha^2 + 5\alpha - 3 = 0$ (1)

$\beta^3 - 6\beta^2 + 5\beta - 3 = 0$ (2)

$\gamma^3 - 6\gamma^2 + 5\gamma - 3 = 0$ (3)

$\alpha^3 + \beta^3 + \gamma^3 = 6(\alpha^2 + \beta^2 + \gamma^2) - 5(\alpha + \beta + \gamma) + 9$

$= 6(26) - 5(6) + 9$

$= 135$ (3)

iii) $\Sigma\alpha^2\beta = \alpha^2\beta + \alpha^2\gamma + \beta^2\alpha + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta$

Now $(\alpha+\beta+\gamma)(\alpha\beta + \alpha\gamma + \beta\gamma) = \alpha^2\beta + \alpha^2\gamma + \alpha\beta\gamma + \beta^2\alpha + \alpha\beta\gamma$
$+ \beta^2\gamma + \alpha\beta\gamma + \alpha\gamma^2 + \beta\gamma^2$

$6 \times 5 = \Sigma\alpha^2\beta + 3\alpha\beta\gamma$

$30 = \Sigma\alpha^2\beta + 3(3)$

$\Sigma\alpha^2\beta = 21$ (3)

d) $8x^4 + 44x^3 + 54x^2 + 25x + 4 = 0$

$32x^3 + 132x^2 + 108x + 25 = 0$

$96x^2 + 264x + 108 = 0$

$8x^2 + 22x + 9 = 0$

$(2x+1)(4x+9) = 0$

$x = -\frac{1}{2}$ or $x = \frac{-9}{4}$

no solution for multiple root

$\therefore 8x^4 + 44x^3 + 54x^2 + 25x + 4 = 0$

$(2x+1)^3(ax+b) = 0$

Equating coefficients $a = 1$ $b = 4$

$(2x+1)^3(x+4) = 0$

$\therefore$ roots are $-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -4$ (4)

e)

i) $P(x) = 2x^3 - x^2 + 2x + 6$

Let $y = x - 1$, since $x = \alpha, \beta, \gamma$

$\therefore x = y + 1$

$$2(y+1)^3 - (y+1)^2 + 2(y+1) + 6 = 0$$
$$2(y^3 + 3y^2 + 3y + 1) - (y^2 + 2y + 1) + 2y + 2 + 6 = 0$$
$$2y^3 + 6y^2 + 6y + 2 - y^2 - 2y - 1 + 2y + 8 = 0$$
$$2y^3 + 5y^2 + 6y + 9 = 0$$

i.e. $2x^3 + 5x^2 + 6x + 9 = 0$ (3)

ii) Let $y = x^2$, since $x = \alpha, \beta, \gamma$

$\therefore y^{\frac{1}{2}} = x$

$$2(y^{\frac{1}{2}})^3 - (y^{\frac{1}{2}})^2 + 2y^{\frac{1}{2}} + 6 = 0$$
$$2y^{3/2} - y + 2y^{\frac{1}{2}} + 6 = 0$$
$$2y^{\frac{1}{2}}(y+1) = y - 6$$
$$4y(y+1)^2 = (y-6)^2$$
$$4y(y^2 + 2y + 1) = y^2 - 12y + 36$$
$$4y^3 + 8y^2 + 4y = y^2 - 12y + 36$$
$$4y^3 + 7y^2 + 16y - 36 = 0$$

i.e. $4x^3 + 7x^2 + 16x - 36 = 0$ (3)

f) Roots are $\alpha, \beta, x+i, x-i$

$$(x+i)(x-i) = x^2 - i^2$$
$$= x^2 + 1$$

$$x^4 + 2x^3 - 2x^2 + 2x - 3 = (x^2+1)(ax^2 + bx + c)$$

Equating coefficients: $a = 1$

$b = 2$

$c = -3$

$$\therefore (x^2+1)(x^2 + 2x - 3) = 0$$
$$(x+i)(x-i)(x+3)(x-1) = 0$$

Roots are $\pm i, -3, 1$ (4)

g) $x^2 + px + q = 0$ and $x^2 + rx + s = 0$

Let the common root be α

$\alpha^2 + p\alpha + q = 0$ (1)

$\alpha^2 + r\alpha + s = 0$ (2)

(1) − (2) $\quad (p-r)\alpha + q - s = 0$

$$\alpha = \frac{s-q}{p-r}$$

Sub for α in (1)

$$\left(\frac{s-q}{p-r}\right)^2 + p\left(\frac{s-q}{p-r}\right) + q = 0$$

$$(s-q)^2 + p(s-q)(p-r) + q(p-r)^2 = 0$$ (2)

Question three

a) $\int_{-3}^{3} (\cos^5 x \sin^5 x)\,dx$

$= 0$

Now $\cos^5 x \sin^5 x$ odd since $(\cos(-x))^5 \sin(-x)^5 = \cos^5 x(-\sin^5 x) = -\cos^5 x \sin^5 x$ /2

b) $\int_{-3}^{3} \sqrt{9-x^2} = \frac{\pi}{2} 3^2 = \frac{9\pi}{2}$ Fn even /2

c) $I = \int \frac{\sin^3 x}{\cos^5 x}\,dx$

$I = \int \tan^3 x \sec^2 x\,dx$

let $u = \tan x$

$du = \sec^2 x\,dx$

$I = \int u^3\,du$

$= \frac{1}{4} u^4$

$= \frac{1}{4}\tan^4 x + C$

Other methods can be used

[Answer must be in terms of x]

or $I = \int \frac{(1-\cos^2 x)\sin x\,dx}{\cos^5 x}$

let $u = \cos x\,dx$

$du = -\sin x\,dx$

$I = -\int \frac{1-u^2}{u^5}\,du$

$= -\int u^{-5} - u^{-3}\,du$

$= -\left(\frac{u^{-4}}{-4} + \frac{u^{-2}}{2}\right)$

$= \frac{u^{-4}}{4} - \frac{u^{-2}}{2}$

$= \frac{1}{4\cos^4 x} - \frac{1}{2\cos^2 x}$

$= \frac{1 - 2\cos^2 x}{4\cos^4 x} + C$

d) i) let $u = \sin ax$ $\quad du = a\cos ax\,dx$

$I = \frac{1}{a}\int u^n\,du$ ①

$= \frac{1}{a}\frac{u^{n+1}}{n+1}$ $(n \neq -1)$ [if $n=-1$ $\int u^{-1} = \ln|u| + c$

$= \frac{1}{a}\frac{\sin^{n+1}(ax)}{n+1}$

ii) let $n = -1$

$I = \int \frac{\cos ax}{\sin ax}\,dx$

$= \int \cot ax$ but from ① above

$= \frac{1}{a}\int u^{-1}\,du$

$= \frac{1}{a}\log_e|u|$

or $\int \cot ax\,dx = \int \frac{\cos ax\,dx}{\sin ax}$

$= \frac{1}{a}\int \frac{a\cos ax}{\sin ax}\,dx$

[in form $\int \frac{f'(x)}{f(x)}\,dx = \log f(x)$]

$= \frac{1}{a}\log_e|\sin ax| + C$

Q3 cont

e) $P(x)$ even

$\therefore$ of form $A(x+1)(x-1)(x+2)^2(x-2)^2$

when $x = 3$, $y = 150$

ie $A(3+1)(3-1)(3+2)^2(3-2)^2 = 150$

$A(4)(2)(25)(1) = 150$

$200A = 150$

$A = \frac{3}{4}$

$\therefore P(A) = \frac{3}{4}(x+1)(x-1)(x+2)^2(x-2)^2$

Note Answer is of degree 6 /3

f) $I = \int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2}$

$= \int \frac{\sin\theta\,d\theta}{(\cos\theta+2)(\cos\theta-1)}$

let $u = \cos\theta$, $du = -\sin\theta\,d\theta$.

$I = -\int \frac{du}{(u+2)(u-1)}$

$\frac{1}{(u+2)(u-1)} = \frac{A}{(u+2)} + \frac{B}{(u-1)}$

$\therefore 1 = A(u-1) + B(u+2)$

put $u = 1$: $1 = 3B \Rightarrow B = \frac{1}{3}$, $\quad u = -2$: $1 = -3A \Rightarrow A = \frac{-1}{3}$

$I = -\int \frac{-1}{3(u+2)} + \frac{1}{3(u-1)}\,du$

$= -\left[-\frac{1}{3}\log_e|u+2| + \frac{1}{3}\log_e|u-1|\right]$

$= \frac{1}{3}\log_e|\cos\theta+2| - \frac{1}{3}\log_e|\cos\theta-1| + C$ /6

$= \frac{1}{3}\log_e\left|\frac{\cos\theta+2}{\cos\theta-1}\right|$

Quickest Method

$I = -\int \frac{\cos\theta\,d\theta}{\cos^2\theta + \cos\theta - 2}$

$= -\int \frac{d\theta}{(\cos\theta+\frac{1}{2})^2 - (\frac{3}{2})^2}$

$= -\frac{1}{2} \times \frac{2}{3}\ln\left|\frac{\cos\theta-1}{\cos\theta+2}\right| + C$

$= -\frac{1}{3}\ln\left|\frac{\cos\theta-1}{\cos\theta+2}\right| + C$

Other methods and answers are possible