Sydney Girls High School

2009 Assessment Task 3

MATHEMATICS

Extension Two

Year 12

Time allowed: 90 minutes (plus 5 minutes reading time)

Topics: Polynomials and Integration

Instructions:

- Attempt all 3 questions
- All necessary working should be shown in every question
- Marks may be deducted for careless or badly arranged work
- Write on one side of the paper only
- A table of standard integrals is supplied

Marks

Question 1. (30 marks)

(a) Find $\int \cos x.\sin x dx$ 2

(b) Find $\int \sin x.\cos^3 x dx$ 4

(c) Find $\int \frac{dx}{\sqrt{(x+3)^2-16}}$ 2

(d) Find $\int 6\sec^2 2x \tan 2x dx$ 3

(e) Find $\int x\cos x dx$ 3

(f) Find $\int \tan^3\theta .d\theta$ 4

(g) Find $\int \frac{dx}{2+3\sin x}$ 4

(h) Find $\int \frac{dx}{x^2\sqrt{x^2-9}}$ 4

(i) It can be shown that $\frac{2(x^2+1)(2x-3)}{(x^2-3)(x^2-4x+5)} = \frac{2x}{x^2-3} + \frac{2x+2}{x^2-4x+5}$ 4

(Do not prove this)

Use this result to find $\int \frac{2(x^2+1)(2x-3)}{(x^2-3)(x^2-4x+5)} dx$

Question 2 (30 marks) Marks

(a) If the polynomial $2x^3-4x^2+4x-3=0$ has roots α,β,γ, find the polynomial with roots:

i. $2\alpha,2\beta,2\gamma$ 2

ii. $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ 3

iii. $\alpha^2,\beta^2,\gamma^2$ 3

iv. $\frac{1}{\alpha\beta},\frac{1}{\alpha\gamma},\frac{1}{\beta\gamma}$ 3

(b) If α,β,γ, are the roots of $3x^3+2x^2-x+1=0$ find

i. $\alpha^2+\beta^2+\gamma^2$ 3

ii. $\alpha^3+\beta^3+\gamma^3$ 2

iii. $\alpha^4+\beta^4+\gamma^4$ 3

iv. $\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2$ 4

(c) (i) Suppose that α is a double root of the polynomial function $P(x)=0$. Show that $P'(\alpha)=0$ 2

(ii) The polynomial $P(x)=ax^3+bx^2+2$ is divisible by $(x+1)^2$
Find the coefficients a and b 3

(iii) Prove that $R(x)=1+x+\frac{x^2}{2}+\frac{x^3}{6}$ has no double roots. 2

Question 3 (30 marks) Marks

(a) If $I_n=\int x^n e^{ax}dx$

i. Show that $I_n=\frac{x^n e^{ax}-nI_{n-1}}{a}$ 3

ii. hence, evaluate $\int_0^1 x^3e^{2x}dx$ 3

(b) If two of the roots of $x^3+px^2+r=0$ are equal, show that $4p^3=-27r$ 3

(c) Find $\int\frac{x.dx}{\sqrt{x^2+6x+5}}$ 4

(d) i. If $P(x)=x^3+x^2-4x+6$ show that $P(1+i)=0$ 1

ii. Hence or otherwise find all three solutions of $P(x)=0$ 4

(e) i. Show that $\int_0^\pi\frac{\sin x}{1+\cos^2 x}dx=\frac{\pi}{2}$ 3

ii. By making the substitution $x=\pi-u$ find $\int_0^\pi\frac{x\sin x}{1+\cos^2 x}dx$ 4

(f) i. The function $x^4+4x^3-2x^2-ax-b$ has two double zeros α and β.
Find the values of α and β. 3

ii. Hence, determine the equation of a straight line that is tangential to the curve $x^4+4x^3-2x^2$ at two distinct points. 2

End of Exam

Solutions Extension 2

Task 3 2009

Question One

a) $I = \int \cos x \sin x \, dx$

let $u = \sin x \Rightarrow du = \cos x \, dx$

$I = \int u \, du$

$= \frac{1}{2} u^2$

$= \frac{1}{2} \sin^2 x + c$ (other solutions are possible) $/2$

b) $I = \int \sin x \cos^3 x \, dx$

let $u = \cos x \Rightarrow du = -\sin x \, dx$

$I = -\int u^3 \, du$ $/4$

$= -\frac{1}{4} \cos^4 x + C$ or $\frac{1}{2}\sin^2 x - \frac{1}{4}\sin^4 x + c$

c) $I = \int \frac{dx}{\sqrt{(x+3)^2 - 16}}$ let $u = x + 3$, $du = dx$

$= \int \frac{du}{\sqrt{u^2 - 16}}$

$= \log_e |u + \sqrt{u^2 - 16}|$

$= \log_e | x + 3 + \sqrt{(x+3)^2 - 16} | + C$ $/2$

d) $I = \int 6 \sec^2 2x \tan 2x \, dx$

let $u = \tan 2x \Rightarrow du = 2 \sec^2 2x \, dx$

$I = 3 \int u \, du$

$= \frac{3u^2}{2}$

$= \frac{3}{2} \tan^2 2x + C$ (other answers are possible) $/3$

e) $I = \int x \cos x \, dx$

let $u = x$, $v' = \cos x$, $u' = 1$, $v = \sin x$

$I = x \sin x - \int \sin x \, dx$

$= x \sin x + \cos x + C$ $/3$

f) $\int \tan^3 x \, dx = \int (\sec^2 x - 1) \tan x \, dx$

$= \int \sec^2 x \tan x \, dx - \int \tan x \, dx$

$= I_1 - I_2$

I_1 let $u = \tan x$, $du = \sec^2 x$

$I = \int u \, du$

$= \frac{1}{2} u^2$

$= \frac{1}{2} \tan^2 x$ (other answers possible)

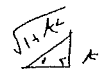
I_2 $\int \tan x \, dx = -\int \frac{-\sin x}{\cos x} dx$

$= -\log_e |\cos x|$

$\therefore I = \frac{1}{2} \tan^2 x + \log_e |\cos x| + C$ $/4$

g) $I = \int \frac{dx}{2 + 3 \sin x}$

let $t = \tan \frac{x}{2}$



$\Rightarrow \frac{dt}{dx} = \frac{1}{2} \sec^2 \frac{x}{2}$

$dx = \frac{2 \, dt}{1 + t^2}$

$\sin x = \frac{2t}{1 + t^2}$

$I = \int \frac{\frac{2 \, dt}{1+t^2}}{2 + 3 \frac{2t}{1+t^2}} \times \frac{1+t^2}{1+t^2}$

$= \int \frac{2 \, dt}{2t^2 + 6t + 2}$

$t^2 + 3t + 1 = t^2 + 3t + (\frac{3}{2})^2 - \frac{5}{4}$

$= (t + \frac{3}{2})^2 - (\frac{\sqrt{5}}{2})^2$

$= \int \frac{dt}{t^2 + 3t + 1}$

$= \int \frac{dt}{(t + \frac{3}{2})^2 - (\frac{\sqrt{5}}{2})^2}$ must complete square correctly

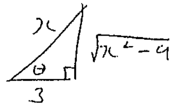
$= \frac{1}{2} \times \frac{2}{\sqrt{5}} \log_e \left| \frac{t + \frac{3}{2} - \frac{\sqrt{5}}{2}}{t + \frac{3}{2} + \frac{\sqrt{5}}{2}} \right|$

$\frac{1}{\sqrt{5}} \log_e \left| \frac{\tan \frac{x}{2} + \frac{3}{2} - \frac{\sqrt{5}}{2}}{\tan \frac{x}{2} + \frac{3}{2} - \frac{\sqrt{5}}{2}} \right| + C$ $/4$

h) $I = \int \frac{dx}{x^2\sqrt{x^2-9}}$ let $x = 3\sec\theta \Rightarrow \frac{x}{3} = \sec\theta$

$dx = 3\sec\theta\tan\theta\,d\theta$

$= \int \frac{3\sec\theta\tan\theta\,d\theta}{9\sec^2\theta\sqrt{9\sec^2\theta-9}}$



$= \int \frac{3\sec\theta\tan\theta\,d\theta}{27\sec^2\theta\tan\theta}$

$= \frac{1}{9}\int \cos\theta\,d\theta$

$= \frac{1}{9}\sin\theta$

$= \frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C$

4

i) $\int \frac{2(x^2+1)(2x-3)}{(x^2-3)(x^2-4x+5)}\,dx = \int \frac{2x}{x^2-3} + \frac{2x+2}{x^2-4x+5}\,dx$

$= \int \frac{2x}{x^2-3} + \frac{2x-4}{x^2-4x+5} + \frac{6}{x^2-4x+5}$

$= \int \frac{2x}{x^2-3} + \frac{2x-4}{x^2-4x+5} + \frac{6}{(x-2)^2+1}\,dx$

$= \log_e|x^2-3| + \log_e|x^2-4x+5| + 6\tan^{-1}(x-2) + C$

4

3(a) i. let $u = x^n$ $v' = e^{ax}$

$u' = nx^{n-1}$ $v = \frac{e^{ax}}{a}$

$I_n = \frac{x^n e^{ax}}{a} - \int nx^{n-1}\frac{e^{ax}}{a}\,dx$

$= \frac{x^n e^{ax} - n\int x^{n-1}e^{ax}\,dx}{a}$

$= \frac{x^n e^{ax} - nI_{n-1}}{a}$

ii $I_3 = \frac{1}{2}\left([x^3e^{2x}]_0^1 - 3I_2\right)$

$= \frac{1}{2}(e^2 - 3I_2)$

$= \frac{e^2}{2} - \frac{3}{2}I_2$

$= \frac{e^2}{2} - \frac{3}{2}\left(\frac{e^2-2I_1}{2}\right)$

$= -\frac{e^2}{4} + \frac{3}{2}I_1$

$= -\frac{e^2}{4} + \frac{3}{2}\left(\frac{e^2 - I_0}{2}\right)$

$= \frac{e^2}{2} - \frac{3}{4}\int_0^1 e^{2x}\,dx$

$= -\frac{e^2}{2} - \frac{3}{4}\left[\frac{e^{2x}}{2}\right]_0^1$

$= -\frac{e^2}{2} - \frac{3e^2}{8} + \frac{3}{8}$

$= \frac{3 + e^2}{8}$

(b) $3x^2 + 2px = 0$

$x(3x + 2p) = 0$

$3x = -2p$

$x = -\frac{2p}{3}$

$\left(-\frac{2p}{3}\right)^3 + p\times\left(-\frac{2p}{3}\right)^2 + r = 0$

$-\frac{8p^3}{27} + \frac{4p^3}{9} + r = 0$

$-8p^3 + 12p^3 + 27r = 0$

$4p^3 = -27r$

(c) $\frac{1}{2}\int \frac{2x+6}{\sqrt{x^2+6x+15}}\,dx - \int \frac{3}{\sqrt{x^2+6x+15}}\,dx$

let $u = x^2+6x+15$ $= \frac{1}{2}\int \frac{2x+6}{\sqrt{u}}\times\frac{du}{2x+6} - \int \frac{3}{\sqrt{(x+3)^2+6}}\,dx$

$\frac{du}{dx} = 2x+6$

$\frac{du}{2x+6} = dx$ $= u^{\frac{1}{2}} - 3\log\left(x+3+\sqrt{x^2+6x+15}\right) + c$

$= \sqrt{x^2+6x+15} - 3\log\left(x+3+\sqrt{x^2+6x+15}\right) + c$

(d) i. $P(1+i) = (1+i)^3 + (1+i)^2 - 4(1+i) + 6$

$(1+i)^2 = 1+2i-1$ $= -2+2i+2i-4-4i+6$

$= 2i$ $= 0$

$(1+i)^3 = 2i + 2i^2$

$= -2+2i$

ii. $1+i$ and $1-i$ are solutions

$1+i+1-i+\alpha = -1$

$2+\alpha = -1$

$\therefore \alpha = -3$

(e) i let $u = \cos x$ $\int_1^{-1} \frac{\sin x}{1+u^2}\times\frac{-du}{\sin x}$

$\frac{du}{dx} = -\sin x$ $= [\tan^{-1}u]_{-1}^{1}$

$-\frac{du}{\sin x} = dx$ $= \frac{\pi}{4} - -\frac{\pi}{4}$

$= \frac{\pi}{2}$

ii $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = -\int_{\pi}^{0} \frac{(\pi-u)\sin(\pi-u)}{1+\cos^2(\pi-u)}\,du$

$= \int_0^{\pi} \frac{(\pi-u)\sin u}{1+\cos^2 u}\,du$

$= \int_0^{\pi} \frac{\pi\sin u}{1+\cos^2 u}\,du - \int_0^{\pi}\frac{u\sin u}{1+\cos^2 u}$

$2\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^{\pi}\frac{\pi\sin u}{1+\cos^2 u}\,du$

$\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,dx = \frac{1}{2}\times\pi\times\frac{\pi}{2}$

$= \frac{\pi^2}{4}$

(f) i $2\alpha + 2\beta = -4$ $-\frac{3}{\beta} + \beta = -2$

$\alpha + \beta = -2$ $-3 + \beta^2 = -2\beta$

$\alpha^2 + 4\alpha\beta + \beta^2 = -2$ $\beta^2 + 2\beta - 3 = 0$

$(\alpha+\beta)^2 + 2\alpha\beta = -2$ $(\beta+3)(\beta-1) = 0$

$(-2)^2 + 2\alpha\beta = -2$ $\alpha = 1, \beta = -3$

$2\alpha\beta = -6$

$\alpha\beta = -3$

$\alpha = -\frac{3}{\beta}$

ii $-b = \alpha^2\beta^2$

$= (-3)^2$

$= 9$

$b = -9$

$-a = -(\alpha^2\beta + \alpha^2\beta + \alpha\beta^2 + \alpha\beta^2)$

$a = 2\alpha\beta(\alpha+\beta)$

$= 2\times -3\times -2$

$= 12$

$\therefore$ eqn is $y = 12x - 9$

Question 2:

a) i) $x = 2\alpha, 2\beta, 2\gamma$

$\frac{x}{2} = \alpha, \beta, \gamma$ (1)

$2\left(\frac{x}{2}\right)^3 - 4\left(\frac{x}{2}\right)^2 + 4\left(\frac{x}{2}\right) - 3 = 0$

$\frac{x^3}{4} - x^2 + 2x - 3 = 0$

$x^3 - 4x^2 + 8x - 12 = 0$ (1)

ii) $x = \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

$\frac{1}{x} = \alpha, \beta, \gamma$ (1)

$2\left(\frac{1}{x}\right)^3 - 4\left(\frac{1}{x}\right)^2 + 4\left(\frac{1}{x}\right) - 3 = 0$

$\frac{2}{x^3} - \frac{4}{x^2} + \frac{4}{x} - 3 = 0$ (1)

$2 - 4x + 4x^2 - 3x^3 = 0$

$3x^3 - 4x^2 + 4x - 2 = 0$ (1)

iii) $x = \alpha^2, \beta^2, \gamma^2$

$\sqrt{x} = \alpha, \beta, \gamma$ (1)

$2(\sqrt{x})^3 - 4(\sqrt{x})^2 + 4\sqrt{x} - 3 = 0$

$2x\sqrt{x} + 4\sqrt{x} - 4x - 3 = 0$

$2\sqrt{x}\,(x+2) = 4x + 3$ (1)

$4x(x+2)^2 = (4x+3)^2$

$4x(x^2+4x+4) = 16x^2 + 24x + 9$

$4x^3 + 16x^2 + 16x = 16x^2 + 24x + 9$

$4x^3 - 8x - 9 = 0$ (1)

iv) $x = \frac{1}{\alpha\beta}, \frac{1}{\alpha\gamma}, \frac{1}{\beta\gamma}$

$= \frac{\gamma}{\alpha\beta\gamma}, \frac{\beta}{\alpha\beta\gamma}, \frac{\alpha}{\alpha\beta\gamma}$ (1)

$\alpha\beta\gamma x = \alpha, \beta, \gamma$ $\qquad \alpha\beta\gamma = \frac{3}{2}$

$\frac{3x}{2} = \alpha, \beta, \gamma$ (1)

$2\left(\frac{3x}{2}\right)^3 - 4\left(\frac{3x}{2}\right)^2 + 4\left(\frac{3x}{2}\right) - 3 = 0$

$2\left(\frac{27x^3}{8}\right) - 4\left(\frac{9x^2}{4}\right) + 6x - 3 = 0$

$\frac{27x^3}{4} - 9x^2 + 6x - 3 = 0$

$27x^3 - 36x^2 + 24x - 12 = 0$ (1)

$9x^3 - 12x^2 + 8x - 4 = 0$

b) $3x^3 + 2x^2 - x + 1 = 0$

i) $\alpha^2 + \beta^2 + \gamma^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$ (1)

$$= \left(-\frac{2}{3}\right)^2 - 2\left(-\frac{1}{3}\right)$$ (1)

$$= \frac{4}{9} + \frac{2}{3}$$

$$= \frac{10}{9}$$ (1)

ii) $3\alpha^3 + 2\alpha^2 - \alpha + 1 = 0$

$3\beta^3 + 2\beta^2 - \beta + 1 = 0$

$3\gamma^3 + 2\gamma^2 - \gamma + 1 = 0$

$3(\alpha^3 + \beta^3 + \gamma^3) + 2(\alpha^2 + \beta^2 + \gamma^2) - (\alpha + \beta + \gamma) + 3 = 0$ (1)

$$3(\alpha^3 + \beta^3 + \gamma^3) = -2\left(\frac{10}{9}\right) + \left(\frac{-2}{3}\right) - 3$$

$$\alpha^3 + \beta^3 + \gamma^3 = \frac{-53}{27}$$ (1)

iii) $3\alpha^4 + 2\alpha^3 - \alpha^2 + \alpha = 0$

$3\beta^4 + 2\beta^3 - \beta^2 + \beta = 0$ (1)

$3\gamma^4 + 2\gamma^3 - \gamma^2 + \gamma = 0$

$3(\alpha^4 + \beta^4 + \gamma^4) + 2(\alpha^3 + \beta^3 + \gamma^3) - (\alpha^2 + \beta^2 + \gamma^2) + (\alpha + \beta + \gamma) = 0$ (1)

$$3(\alpha^4 + \beta^4 + \gamma^4) + 2\left(\frac{-53}{27}\right) - \left(\frac{10}{9}\right) + \left(\frac{-2}{3}\right) = 0$$

$$3(\alpha^4 + \beta^4 + \gamma^4) - \frac{154}{27} = 0$$

$$\alpha^4 + \beta^4 + \gamma^4 = \frac{154}{81}$$ (1)

iv) $(\alpha\beta + \alpha\gamma + \beta\gamma)^2 = \alpha\beta(\alpha\beta + \alpha\gamma + \beta\gamma)$
$+ \alpha\gamma(\alpha\beta + \alpha\gamma + \beta\gamma)$
$+ \beta\gamma(\alpha\beta + \alpha\gamma + \beta\gamma)$ (1)

$= \alpha^2\beta^2 + \alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha^2\beta\gamma + \alpha^2\gamma^2 + \alpha\beta\gamma^2$
$+ \alpha\beta^2\gamma + \alpha\beta\gamma^2 + \beta^2\gamma^2$

$= (\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2) + 2\alpha^2\beta\gamma + 2\alpha\beta^2\gamma + 2\alpha\beta\gamma^2$ (1)

$= (\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2) + 2\alpha\beta\gamma(\alpha + \beta + \gamma)$ (1)

$$\left(-\frac{1}{3}\right)^2 = (\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2) + 2\left(-\frac{1}{3}\right)\left(-\frac{2}{3}\right)$$

$$\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 = -\frac{1}{3}$$ (1)

c)

i) let $P(x) = (x-\alpha)^2 \, Q(x)$

$P'(x) = (x-\alpha)^2 . Q'(x) + Q(x) . 2(x-\alpha)$

$= (x-\alpha)\left[(x-\alpha)Q'(x) + 2Q(x)\right]$ ①

$\therefore (x-\alpha)$ is a factor of $P'(x)$

$\therefore \quad P'(\alpha) = 0$ ①

ii) $P(x) = ax^3 + bx^2 + 2$

$P'(x) = 3ax^2 + 2bx$

$P(-1) = a(-1)^3 + b(-1)^2 + 2$

$0 = -a + b + 2$

$a - b = 2$ ① ①

$P'(-1) = 3a(-1)^2 + 2b(-1)$

$0 = 3a - 2b$ ② ①

From ① $a = b + 2$

Sub into ② $0 = 3(b+2) - 2b$

$= b + 6$

$b = -6$

Sub $b = -6$ into ① $a = -6 + 2$

$= -4$

$\therefore a = -4 \quad b = -6$ ①

iii) $R(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$

$R'(x) = 1 + x + \frac{x^2}{2}$

If $x = \alpha$ is a double root, $R(\alpha) = R'(\alpha) = 0$ ①

$R(\alpha) = 1 + \alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{6} = 0$ ①

$R'(\alpha) = 1 + \alpha + \frac{\alpha^2}{2} = 0$ ②

① − ② $\frac{\alpha^3}{6} = 0$

$\alpha = 0$

but $R(0) = 1$ and $R'(0) = 1$

$\therefore x = 0$ is not a root

$\therefore R(x)$ has no double roots. ①