Sydney Girls' High School

LABOR OMNIA VINCIT

March - 2010
MATHEMATICS - Extension 1

YEAR 12
ASSESSMENT TASK 2

Time Allowed: 75 minutes (plus 5 mins reading time)

Name: __

Teacher: __

TOPICS: Exponential and Logarithmic Functions, the Trigonometric Functions and Trigonometric Function II.

Directions to Candidates

- There are five (5) questions.
- Attempt ALL questions.
- Questions are of equal value.
- Start each question on a new page.
- Write on one side of the paper only.
- Show all necessary working.
- Marks may be deducted for careless or badly arranged work.
- Diagrams are NOT drawn to scale.
- Board-approved calculators may be used.

Total: 75 marks

10:15

QUESTION 1 (15 marks) — Marks

(a) Differentiate the following with respect to x

(i) $\log_e (3x + 1)$ — 1

(ii) $x\sin x$ — 2

(iii) $x\log_e \cos x + \sin^2 x$ — 3

(b) Solve $e^{3t} = 5$ — 2

(c) Find the value of a if $\int_a^e \frac{1}{x}\,dx = 5$ — 3

(d) Find the equation of the normal to the curve $y = \log_e x$ at the point where $x = e^2$. — 4

QUESTION 2 (15 marks)

(a) Find $\lim_{x\to 0} \frac{\sin 2x}{6x}$ — 2

(b) Find the primitive of: (i) $\sqrt{x}$ - e^{-x} — 1

(ii) $\sin x + \sec^2 x$ — 1

(c) Evaluate: (i) $\int_0^{\frac{\pi}{2}} \sin\, 2x\, dx$ — 2

(ii) $\int_3^4 \frac{x-1}{x^2-2x}\,dx$ — 3

(d) Prove $\frac{\sin 2x}{1+\cos 2x} = \tan x$ — 2

(e) If $\tan\alpha = \frac{3}{4}$ and α is acute and $\cos\beta = \frac{-2}{5}$ and β is obtuse, find the exact values of:

(i) $\sin 2\alpha$ — 2

(ii) $\cos\,(\alpha+\beta)$ — 2

QUESTION 3 (15 marks)

Marks

(a) For the curve $y = f(x)$ it is given that $f'(x) = \sin^2 x$.
The curve passes through the point $(\frac{\pi}{4}, \frac{\pi}{8})$.
Find the equation of the curve. 4

(b) Find the acute angle between $x - 2y = 0$ and $3x - y - 15 = 0$ 3

(c)

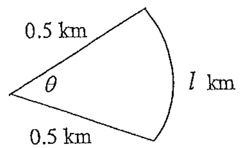


Figure not to scale

A car travels at 40 km/h on a circular curve whose radius is 0.5 km.

(*i*) Find the distance, l km, that the car travels in one minute. 2

(*ii*) Calculate the size of the angle θ through which the car turns in one minute.
Give your answer to the nearest degree. 2

(d) Express $\sqrt{3}\sin x - \cos x$ in the form $R\sin(x - \alpha)$, $R > 0$.
Hence or otherwise, find the solution for $\sqrt{3}\sin x - \cos x = 1$ in the domain $0 \le x \le 2\pi$. 4

QUESTION 4 (15 marks)

Marks

(a) Differentiate xe^x and hence, evaluate $\int_0^1 xe^x\,dx$. 3

(b) Given $y = \sin x$ and $y = \frac{2}{\pi}x$ for $0 \le x \le \pi$.

(*i*) Sketch $y = \sin x$ and $y = \frac{2}{\pi}x$ on the same diagram for $0 \le x \le \pi$. 2

(*ii*) Find their point(s) of intersection. 2

(*iii*) Find the area enclosed between the two curves. 2

(c) (*i*) If $0 \le \theta \le \frac{\pi}{2}$, prove that $\tan\theta = \sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}$ 3

(*ii*) *Hence*, show that the exact value of $\tan\frac{\pi}{8}$ is $\sqrt{2} - 1$. 3

QUESTION 5 (15 marks)

(a) Solve $3t^2 + 4t - 4 = 0$ for $0 \le x \le 2\pi$ where $t = \tan\frac{x}{2}$.
Give answer in radians, correct to two decimal places. 4

(b) Find the smallest positive value of θ if $\cos\theta - \sin 2\theta = 0$. 3

(c) Calculate the volume of the solid generated when the region in the first quadrant bounded by the curve $y = e^x$, the line $x = \log_e 2$ and the co-ordinate axes is rotated about the x-axis. 3

(d) (*i*) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ 2

(*ii*) *Hence,* solve the equation $\cos 3\theta - \cos\theta = 0$, $0^\circ \le \theta \le 360^\circ$. 3

THE END

Yr 12 - Mathematics - Ext 1 - March 2010 - Task 2

Question 1 (15 marks)

i) let $y = \log_e(3x+1)$

$$\frac{dy}{dx} = \frac{3}{3x+1}$$ (1)

ii) let $y = x\sin x$

$u = x, \quad v = \sin x$

$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = \cos x$

$\therefore \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$$\frac{dy}{dx} = \sin x + x\cos x$$ (2)

iii) let $y = x.\log_e \cos x + \sin^2 x$

$y = x.\log_e \cos x + (\sin x)^2$

$\therefore \frac{dy}{dx} = x.\frac{-\sin x}{\cos x} + 1.\ln\cos x$

$2\sin x.\cos x$ (3)

$$\frac{dy}{dx} = \ln\cos x - x.\tan x + 2\sin x\cos x.$$

) $e^{3t} = 5$

$\log_e e^{3t} = \log_e 5$

$3t.\log_e e = \log_e 5$

$3t = \log_e 5$

$$t = \frac{\log_e 5}{3}$$ (2)

c) $\int_a^e \frac{1}{x}\,dx = 5$

$\ln e - \ln a = 5$

$1 - \ln a = 5$

$\ln a = -4$ (3)

$\therefore a = e^{-4} = \frac{1}{e^4}$

d) $y = \log_e x \qquad x = e^2$

$\frac{dy}{dx} = \frac{1}{x} \qquad$ at $x = e^2$

$\frac{dy}{dy} = \frac{1}{e^2}$ and $y = \log_e e^2$

$y = 2\log_e e$

since $m_1 = -\frac{1}{m_2} \qquad y = 2$

$\therefore m_2 = -e^2 \qquad$ and

$P(e^2, 2)$

$\therefore$ Equation of normal

$y - y_1 = m(x - x_1)$

$y - 2 = -e^2(x - e^2)$

$y - 2 = -e^2x + e^4$

$\therefore y = -e^2x + e^4 + 2$

(4)

Question 2 (15 marks)

a) $\lim_{x\to 0} \frac{\sin 2x}{6x} = \frac{2}{6}\lim_{x\to 0}\frac{\sin 2x}{2x}$

$= \frac{1}{3}$ (2)

b) i) $\int x^{1/2} - e^{-x}\,dx$

$= \frac{2}{3}x^{3/2} + e^{-x} + C$ (1)

ii) $\int \sin x + \sec^2 x\,dx$

$= -\cos x + \tan x + C$ (1)

c) i) $\int_0^{\pi/2} \sin 2x\,dx$

$= \left[-\frac{\cos 2x}{2}\right]_0^{\pi/2}$

$= -\frac{1}{2}\left[\cos\pi - \cos 0\right]$

$= -\frac{1}{2} \times -2$ (2)

$= 1$

ii) $\int_3^4 \frac{x-1}{x^2-2x}\,dx$

$= \frac{1}{2}\log_e(x^2-2x)\Big]_3^4$

$= \frac{1}{2}\left[\log_e(16-8) - \log_e(9-6)\right]$

$= \frac{1}{2}\left[\log_e 8 - \log_e 3\right]$ (3)

$= \frac{1}{2}\log_e 8/3$

d) Prove

$\frac{\sin 2x}{1+\cos 2x} = \tan x$

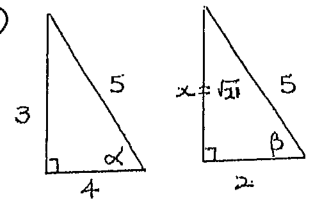
$L.H.S = \frac{2\sin x\cos x}{2\cos^2 x}$

$= \frac{\sin x}{\cos x}$

$= \tan x$ (2)

$= R.H.S.$

e)



$x^2 = 25 - 4$

$x = \sqrt{21}$

i) $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2 \times \frac{3}{5} \times \frac{4}{5}$

$= \frac{24}{25}$ (2)

ii) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \frac{4}{5} \times \frac{2}{5} - \frac{3}{5} \times \frac{\sqrt{21}}{5}$

$= \frac{8}{25} - \frac{3\sqrt{21}}{25}$

$= \frac{8 - 3\sqrt{21}}{25}$ (2)

Question 3 (15 marks) - Ext 1

$f'(x) = \sin^2 x$

$= \frac{1}{2}(1 - \cos 2x)$

$f(x) = \int \frac{1}{2}(1 - \cos 2x)\,dx$

$= \frac{1}{2}(x - \frac{1}{2}\sin 2x) + C$

$f(\frac{\pi}{4}) = \frac{\pi}{8}$

$\therefore \frac{\pi}{8} = \frac{1}{2}(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}) + C$

$\frac{\pi}{8} = \frac{\pi}{8} - \frac{1}{4} + C$

$\therefore C = \frac{1}{4}$

$f(x) = \frac{1}{2}(x - \frac{1}{2}\sin 2x) + \frac{1}{4}$

OR/ $f(x) = \frac{1}{2}x - \frac{1}{4}\sin 2x + \frac{1}{4}$

(4)

) $x - 2y = 0$ $\qquad$ $3x - y - 15 = 0$

$-2y = -x$ $\qquad$ $-y = -3x + 15$

$y = \frac{1}{2}x$ $\qquad$ $y = 3x - 15$

$\therefore m_1 = \frac{1}{2}$ $\qquad$ $\therefore m_2 = 3$

$\therefore \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$\tan\theta = \left|\frac{\frac{1}{2} - 3}{1 + \frac{1}{2}\cdot 3}\right|$

$\tan\theta = 1$

$\therefore \theta = 45°$

(3)

c) i) $d = \frac{s}{t}$

$d = \frac{40\text{km/h}}{60}$

$= \frac{2}{3}$ km in 1 minute

$\therefore l = 2/3$ km

(2)

ii) $\theta = \frac{l}{r}$

$= \frac{\frac{2}{3}}{0.5}$

$\theta = \frac{4}{3}$ radians

$\therefore \theta = \frac{4}{3} \times \frac{180}{\pi}$

$= 76°.$

(2)

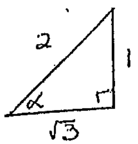
d) $R = \sqrt{(\sqrt{3})^2 + 1} = 2$

$\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x = \cos\alpha \sin x - \sin\alpha\cos x$

where $\cos\alpha = \frac{\sqrt{3}}{2}$ and

$\sin\alpha = \frac{1}{2}$.

and α is acute



$\alpha = \frac{\pi}{6}.$

Now, $\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x = \frac{1}{2}$

$\sin(x - \alpha) = \frac{1}{2}$

$x - \alpha = \frac{\pi}{6}, \pi - \frac{\pi}{6}$

$x - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore x = \frac{\pi}{3}$ and π.

(4)

Question 4 (15 marks) - Ext 1

a) let $y = xe^x$

$\frac{dy}{dx} = e^x + xe^x$

$\therefore \int_0^1 xe^x\,dx = \int_0^1 e^x + xe^x\,dx - \int_0^1 e^x\,dx$

$= \left[xe^x - e^x\right]_0^1$

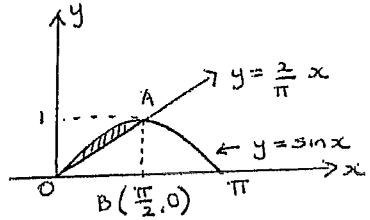
$= \left[(1.e^1 - e^1) - (0 - e^0)\right]$

$= [0 - -1]$

$= 1$

(3)

b) i)



(2)

ii) Points of intersection

$P_1(0, 0)$

$P_2(\frac{\pi}{2}, 0)$

(2)

iii) Shaded Area $= \int_0^{\pi/2} \sin x\,dx -$ area of ΔAOB

$= \left[-\cos x\right]_0^{\pi/2} - \frac{1}{2}\cdot\frac{\pi}{2}\cdot 1$

$= -\left[\cos\pi/2 - \cos 0\right] - \frac{\pi}{4}$

$= (1 - \pi/4)$ units².

(2)

c) i) $2\cos^2\theta = \cos 2\theta + 1$

$2\sin^2\theta = 1 - \cos 2\theta$

$\tan\theta = \sqrt{\frac{1 - \cos 2\theta}{1 + \cos 2\theta}}$

$R.H.S = \sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}$

$= \sqrt{\tan^2\theta}$

$= \tan\theta$ since $\tan\theta > 0$ for $0 \leq \theta \leq \frac{\pi}{2}$

$= L.H.S$

(3)

ii) substitute $\theta = \pi/8$

$\therefore \tan\pi/8 = \sqrt{\frac{1 - \cos\pi/4}{1 + \cos\pi/4}}$

$= \sqrt{\frac{1 - \frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}}}$

$= \sqrt{\frac{\sqrt{2} - 1}{\sqrt{2} + 1} \times \frac{\sqrt{2} - 1}{\sqrt{2} - 1}}$

$= \sqrt{\frac{(\sqrt{2} - 1)^2}{2 - 1}}$

$= \sqrt{(\sqrt{2} - 1)^2}$

$= \sqrt{2} - 1$

$\therefore \tan\pi/8 = \sqrt{2} - 1$

(3)

Question 5 (15 marks) - Ext 1

a) $t = \tan\frac{x}{2}$

$3t^2 + 4t - 4 = 0$

$(3t-2)(t+2) = 0$

$3t = 2 \qquad t = -2$

$t = 2/3$

$\therefore \tan\frac{x}{2} = \frac{2}{3}$ or $\tan\frac{x}{2} = -2$

$\frac{x}{2} = \tan^{-1}\frac{2}{3} \qquad \frac{x}{2} = \tan^{-1} -2$

$\frac{x}{2} = 0.588 \qquad \frac{x}{2} = \pi - 1.107$

$x = 1.18 \qquad x = 2\pi - 2.2$

$x = 4.07$

$\therefore x = 1.18$ or 4.07

(4)

b) $\cos\theta - \sin 2\theta = 0$

$\sin 2\theta = \cos\theta$

$2\sin\theta\cos\theta = \cos\theta$

$2\sin\theta\cos\theta - \cos\theta = 0$

$\cos\theta(2\sin\theta - 1) = 0$

$\cos\theta = 0$ or $2\sin\theta = 1$

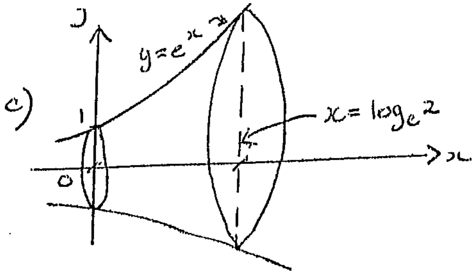
$\sin\theta = \frac{1}{2}$

$\therefore \theta_1 = \frac{\pi}{2}$ or $\frac{3\pi}{2} \qquad \theta_2 = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

$\therefore$ smallest positive value of θ is $\frac{\pi}{6}$.

(3)

c)



Volume of solid generated

$= \pi\int_0^{\log_e 2} y^2\,dx$ where $y = e^x$

$= \pi\int_0^{\log_e 2} e^{2x}\,dx$

$= \frac{\pi}{2}\left[e^{2x}\right]_0^{\log_e 2}$

$= \frac{\pi}{2}\left[e^{2\log_e 2} - e^0\right]$

$= \frac{\pi}{2}[4 - 1]$

$= \frac{3\pi}{2}$ units^3

(3)

d) i) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

L.H.S

$\cos(3\theta) = \cos(2\theta + \theta)$

$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\cos\theta\sin\theta\sin\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta\sin^2\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta)$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

$=$ R.H.S.

(2)

Question 5 (con't)

ii) Solve

$\cos 3\theta - \cos\theta = 0$

$4\cos^3\theta - 3\cos\theta - \cos\theta = 0$

$4\cos^3\theta - 4\cos\theta = 0$

$4\cos\theta(\cos^2\theta - 1) = 0$

$4\cos\theta = 0$ or $\cos^2\theta - 1 = 0$

$\cos\theta = 0 \qquad \cos^2\theta = 1$

$\cos\theta = \pm 1$

$\therefore \theta = 90°$ or $270°$

$\cos\theta = 1$

$\therefore \theta = 0, 360°$

or $\cos\theta = -1$

$\theta = 180°$

$\therefore \theta = 0°, 90°, 180°, 270°$ or $360°$.

(3)