SYDNEY GIRLS HIGH SCHOOL

2006 HSC Assessment Task 3

June 7, 2006

MATHEMATICS

Year 12

Time allowed: 90 minutes

Topics: Locus & Parabola, Quadratics, Integration

DIRECTIONS TO CANDIDATES:

- Attempt all questions
- Questions are of equal value
- Part marks shown in brackets
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work
- Board approved calculators may be used
- Write on one side of the paper only

SGHS Mathematics 2U – Assessment Task 3: June 2006

Question 1.

a) For the parabola $x^2 = 4(y-1)$, write down

   i) The vertex
   ii) The focus
   iii) The equation of the directrix
   iv) The equation of the axis of symmetry [4]

b) A parabola has the point (2, -2) as its vertex and (2,0) as the focus. Write down the equation of the parabola. [2]

c) Find the locus of the set of points P(x,y) given A(-1,-1) and B(5,3), such that

   i) PA and PB are the same length
   ii) PA and PB are perpendicular. [4]

d) For the parabola $x^2 = 16y$,

   i) Find the gradient at the point P(-8,4).
   ii) Hence find the equations of the tangent and normal at P.
   iii) If the tangent cuts the Y axis at M and the normal cuts the Y axis at N, find the area of triangle PMN. [6]

e) Find the locus of the set of points where P(x,y) is equidistant from A(2,3) and the line y = 1 [4]

Question 2.

a) Solve for x where: $x^2 - 7x - 18 = 0$ [2]

b) For what values of k is the expression $x^2 - 2(k-3)x + (k-1)$ positive definite?

[4]

c) If α and β are the roots of the equation $2x^2 - 7x + 2 = 0$, find the values of:

i) $\alpha + \beta$

ii) $\alpha\beta$

iii) $\frac{1}{\alpha} + \frac{1}{\beta}$

iv) $\alpha^2 + \beta^2$

v) $\frac{4}{\alpha^2} + \frac{4}{\beta^2}$ [7]

d) Find all the real numbers x that satisfy the equation: $x^4 = 4x^2 + 32$

[3]

e) Find the values of k for the function $f(x) = 2x^2 - (3k-1)x + (2k-5) = 0$ to have

i) Sum of roots to be 4

ii) The roots to be reciprocals [4]

Question 3.

a) i) Use the Trapezoidal Rule with 3 values to estimate $\int_0^1 4^x.dx$ [3]

ii) Is the estimate an under estimate or over estimate. Justify your answer. [1]

b) Find the following indefinite integrals:

i) $\int (4-3x)^5.dx$

ii) $\int \frac{x^4-1}{x^2}.dx$

iii) $\int \left(x+\frac{1}{x}\right)^2.dx$

[6]

c) The curve $y=f(x)$ has the gradient function $f'(x)=3x^2-2x+1$. If the curve passes through the point Q(2,3), find the equation of the function. [3]

d) Given the function $y=16^x$

i) Copy and complete the following table

x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
y					

ii) Use two applications of Simpson's Rule to find the approximate area enclosed by the curve, the X axis and the line x = 1 [3]

e) Given $y=\sqrt{x^2+16}$,

i) Find $\frac{dy}{dx}$

ii) Hence or otherwise evaluate $\int_0^3 \frac{2x.dx}{\sqrt{x^2+16}}$ [4]

Question 4.

a) A parabola has the equation $x^2 = 8(4-y)$

i) Sketch the parabola and clearly indicate the directrix, focus and points of intersection with the co-ordinate axes.

ii) Another parabola Q, with equation $x^2 = 8y$ intersects the parabola P at A and B. Find the co-ordinates of A and B

iii) Calculate the area of the region bounded by the two parabolas P and Q: [8]

b) Given $2mx^2 - (4m+1)x + 2 = 0$, show that the equation has rational roots if m is rational.

[3]

c) i) Find the points of intersection of the curve $y = 4 - \sqrt{2x}$ with the X and Y axes.

ii) The area enclosed by the curve $y = 4 - \sqrt{2x}$ and the X and Y axes is rotated about the Y axis. Find the volume of the solid of revolution that is formed.

[5]

d) Consider the points A(-3,-1) and B (6,2). If a point P(x,y) moves so that PA is twice the distance PB, show that the locus of P is a circle and find its centre and radius.

[4]

--end of exam--

YR 12 2U 2006 Q (1)

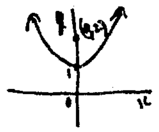
(1)

(a) (i) $(0,1)$ (1)

(ii) $(0,2)$ (1)

(iii) $y=0$ (1)

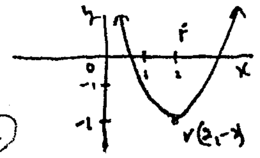
(iv) $x=0$ (1)



(b) $x^2 = 4ay$

$(x-h)^2 = 4a(y-k)$

$(x-2)^2 = 8(y+2)$ (2)



$a = 2$

(c) (i) $\sqrt{(x+1)^2+(y+1)^2} = \sqrt{[illegible]+(y-3)^2}$

$x^2+2x+1+y^2+2y+1 = x^2-10x+25+y^2-6y+9$

$12x+8y-32=0$

$3x+2y-8=0$ (2)

(ii) $m_{PA} \times m_{PB} = -1$

$\frac{y+1}{x+1} \cdot \frac{y-3}{x-5} = -1$

$\frac{y^2-2y-3}{x^2-4x-5} = -1$ (2)

$y^2-2y-3 = -x^2+4x+5$

$x^2-4x+y^2-2y-8=0$

or $(x-2)^2+(y-1)^2=13$

(d) (i) $y = \frac{x^2}{16}$ (1)

$y' = \frac{x}{8}$

at $x=-8$ $y'=-1$

(ii) $y-4 = -1(x+8)$ (3)

$y-4 = -x-8$

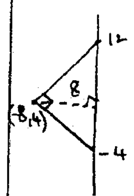
$(y=-x-4)$ $x+y+4=0$ (Tangent)

$y-4 = 1(x+8)$

$y-4 = x+8$

$(y=x+12)$ $x-y+12=0$ (Normal)

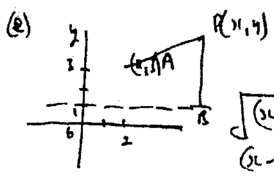
(d) (iii)



$\text{AREA} = \frac{(12--4)(8)}{2}$ (2)

$= 64u^2$

(e) (4)



$PA = PB$

$\sqrt{(x-2)^2+(y-3)^2} = y-1$

$(x-2)^2+(y-3)^2 = (y-1)^2$

$(x-2)^2+y^2-6y+9 = y^2-2y+1$

$(x-2)^2 = 4y-8$

$(x-2)^2 = 4(y-2)$

or $x^2-4x-4y+12=0$

Q2

a) $x^2-7x-18=0$

$(x-9)(x+2)=0$

$x-9=0$ or $x+2=0$

$x=9$ (1) $x=-2$ (1)

b) $\Delta = b^2-4ac$

$= [-2(k-3)]^2 - 4(1)(k-1)$

$= 4(k^2-6k+9)-4k+4$

$= 4k^2-24k+36-4k+4$

$= 4k^2-28k+40$

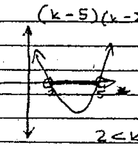
$= 4(k^2-7k+10)$ (2)

pos. def when $\Delta < 0$ and $a > 0$ (1)

$4(k^2-7k+10) < 0$

$k^2-7k+10 < 0$

$(k-5)(k-2) < 0$



$2 < k < 5$ (1)

c) i) $\alpha+\beta = \frac{-b}{a}$

$= \frac{7}{2}$

$= 3\frac{1}{2}$ (1)

ii) $\alpha\beta = \frac{c}{a}$

$= \frac{2}{2}$

$= 1$ (1)

iii) $\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta}$

$= \frac{7}{2} \div 1$

$= \frac{7}{2}$ (1)

iv) $(\alpha^2+\beta^2) = (\alpha+\beta)^2 - 2\alpha\beta$

$= \left(\frac{7}{2}\right)^2 - 2(1)$

$= \frac{49}{4} - 2$

$= \frac{41}{4}$

$= 10\frac{1}{4}$ (2)

v) $\frac{4}{\alpha^2}+\frac{4}{\beta^2} = \frac{4\beta^2+4\alpha^2}{\alpha^2\beta^2}$

$= \frac{4(\alpha^2+\beta^2)}{(\alpha\beta)^2}$

$= \frac{4 \times \frac{41}{4}}{(1)^2}$

$= 41$ (2)

d) $x^4 = 4x^2+32$

$x^4-4x^2-32=0$

Let $m = x^2$

$m^2-4m-32=0$ (3)

$(m-8)(m+4)=0$

$m-8=0$ or $m+4=0$

$m=8$ $m=-4$

$x^2=8$ $x^2=-4$

$x=\pm 2\sqrt{2}$ no solution

e) $2x^2-(3k-1)x+(2k-5)=0$

i) let roots be α, $4-\alpha$.

$\alpha+(4-\alpha) = \frac{3k-1}{2}$

$4 = \frac{3k-1}{2}$

$3k-1 = 8$

$3k = 9$

$k = 3$ (2)

ii) $\alpha\beta = 1$ (reciprocal roots)

$\alpha\beta = \frac{2k-5}{2}$

$1 = \frac{2k-5}{2}$

$2k-5 = 2$

$2k = 7$

$k = 3\frac{1}{2}$ (2)

Question 3

a i $\int_0^1 4^x \, dx$

x	$y = 4^x$	w	$w.y$
0	1	1	1
0.5	2	2	4
1	4	1	4
			$\Sigma wy = 9$

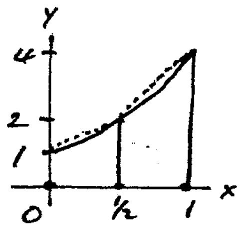
$\int_0^1 4^x \, dx = \frac{h}{2} \times \Sigma wy$

$= \frac{1/2}{2} \times 9$

$= \frac{9}{4}$

$= 2\frac{1}{4}$ (or 2.25) (3)

ii



(1)

From Graph = Overestimate.

or $\int_0^1 4^x \, dx = \left[\frac{4^x}{\log_e 4}\right]_0^1 = \frac{(4-1)}{\log_e 4} = 2.16$

iii $y = 16^x$

x	$y = 16^x$	w	$w.y$
0	1	1	1
$\frac{1}{4}$	2	4	8
$\frac{1}{2}$	4	2	8
$\frac{3}{4}$	8	4	32
1	16	1	16
	↑ (1)		$\Sigma wy = 65$

$\int_0^1 16^x \, dx = \frac{h}{3} \times \Sigma w.y$

$= \frac{1/4}{3} \times 65$

$= \frac{65}{12}$

$= 5\frac{5}{12}$ units² (2)

b i $\int (4-3x)^5 \, dx$ (2)

$= \frac{(4-3x)^6}{-3 \times 6} = \frac{(4-3x)^6}{-18} + c$

ii $\int \frac{x^4 - 1}{x^2} \, dx$ (2)

$= \int x^2 - x^{-2} \, dx$

$= \frac{x^3}{3} - \frac{x^{-1}}{-1} = \frac{x^3}{3} + x^{-1} + c$

iii $\int \left(x + \frac{1}{x}\right)^2 dx$

$= \int x^2 + 2 + x^{-2} \, dx$ (2)

$= \frac{x^3}{3} + 2x - x^{-1} + c$

(or $\frac{x^3}{3} + 2x - \frac{1}{x} + c$)

c $f'(x) = 3x^2 - 2x + 1$

$f(x) = x^3 - x^2 + x + C$

Substitute $(2, 3)$

$3 = 8 - 4 + 2 + C$

$\therefore C = -3$ (3)

$\therefore f(x) = x^3 - x^2 + x - 3$

d $y = \sqrt{x^2 + 16} = (x^2 + 16)^{\frac{1}{2}}$

i $\frac{dy}{dx} = \frac{1}{2} \cdot 2x \, (x^2 + 16)^{-1/2}$

$= \frac{x}{\sqrt{x^2 + 16}}$ (2)

ii $\int_0^3 \frac{2x}{\sqrt{x^2+16}} \, dx$

$= 2\left[\sqrt{x^2+16}\right]_0^3$

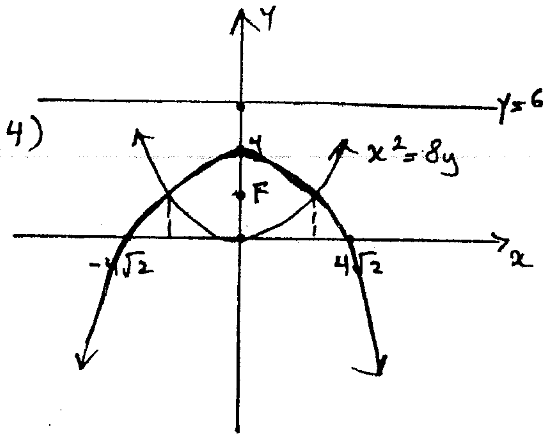
$= 2\left[\sqrt{25} - \sqrt{16}\right]$

$= 2$ (2)

Yr 12 2U 2006
Assessment task 3

Q4.

a) i) $x^2 = -8(y-4)$



$V(0,4)$

$4a = 8$
$a = 2$

$F(0,2)$
directrix $y=6$

(2)

$y=0 \rightarrow x^2 = 32$
$x = \pm 4\sqrt{2}$

ii) $x^2 = 8y$ — ①
$x^2 = 8(4-y)$ — ②
① = ②
$8y = 32 - 8y$
$16y = 32$
$\boxed{y=2}$
sub in ①
$x^2 = 16$
$x = \pm 4$

(3)

$A(4, 2)$
$B(-4, 2)$

$y = \frac{x^2}{8}$
$8y = 32 - x^2$

iii) $A = 2\int_0^4 \frac{32-x^2}{8} - \frac{x^2}{8}\,dx$

$= 2\int_0^4 4 - \frac{x^2}{4}\,dx$

$= 2\left[4x - \frac{x^3}{12}\right]_0^4$

$= 2\left[16 - \frac{64}{12}\right]$

$= 21\frac{1}{3}\ u^2$

(3)

b) For rational
Δ is a perfect square

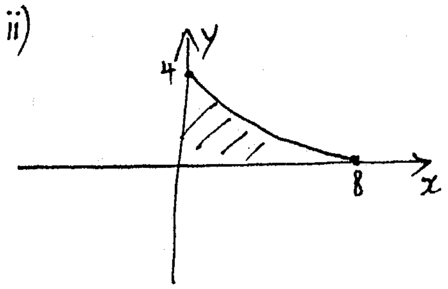
$\Delta = b^2 - 4ac$
$= (-[4m+1])^2 - 4 \times 2m \times 2$
$= 16m^2 + 8m + 1 - 16m$
$= 16m^2 - 8m + 1$

$= (4m-1)^2$ (3)

If m is rational then the roots are rational

c) i) $x = 0$ $y = 4$ ①
when $y = 0$
$4 = \sqrt{2x}$
$2x = 16$
$x = 8$ ①

ii)



$Vol = \pi\int_0^4 \left(\frac{(4-y)^2}{2}\right)^2 dy$

$= \pi\int_0^4 \frac{(4-y)^4}{4}\,dy$

ii) cont

$V = \pi\left[\frac{(4-y)^5}{-5\times 4}\right]_0^4$

$= \pi\left[0 - \frac{1024}{-20}\right]$

$= 51\frac{1}{5}\pi\ u^3$

(3)

d) $PA = 2PB$

$(x+3)^2 + (y+1)^2 = 4[(x-6)^2 + (\ldots$

$x^2+6x+9+y^2+2y+1 = 4[(x^2-12x+36)+(y^2-4y+4\ldots$

$3x^2 - 54x + 150 + 3y^2 - 18y = \ldots$
$x^2 - 18x + y^2 - 6y = -50 + \ldots$
$(x-9)^2 + (y-3)^2 = 40$

$C(9,3)$ $r = 2\sqrt{10}$

(4)