QUESTION ONE (15 marks) Use a separate writing booklet. **Marks**

(a) Find $\int \frac{1}{x^2+4x+7}\,dx$. $\boxed{2}$

(b) (i) Express $\frac{8}{(x-2)(x^2-4x+8)}$ in the form $\frac{A}{x-2}+\frac{Bx+C}{x^2-4x+8}$. $\boxed{3}$

(ii) Hence find $\int \frac{8}{(x-2)(x^2-4x+8)}\,dx$. $\boxed{3}$

(c) Use the substitution $x = 2\sin\theta$ to evaluate $\boxed{4}$

$$\int_0^1 \frac{1}{(4-x^2)^{\frac{3}{2}}}\,dx.$$

(d) Use integration by parts to evaluate $\int x\tan^{-1}x\,dx$. $\boxed{3}$

QUESTION TWO (15 marks) Use a separate writing booklet. **Marks**

(a) Given that $z = 1-2i$, write each of the following in the form $x+iy$, where x and y are real numbers:

(i) $\bar{z} - iz$ $\boxed{2}$

(ii) $\frac{1}{2-z}$ $\boxed{2}$

(b) (i) Express $\sqrt{3}-i$ in modulus–argument form. $\boxed{2}$

(ii) Hence find the values of the real numbers a and b such that $\boxed{2}$

$$\left(\sqrt{3}-i\right)^8 = 2^7(a+ib).$$

(c) Sketch on the Argand diagram the region that simultaneously satisfies: $\boxed{3}$

$$0 \le \arg(z+1) \le \frac{3\pi}{4} \quad \text{and} \quad |z+1| \le 2.$$

$2-(1-2i)$

$= 2-1+2i$

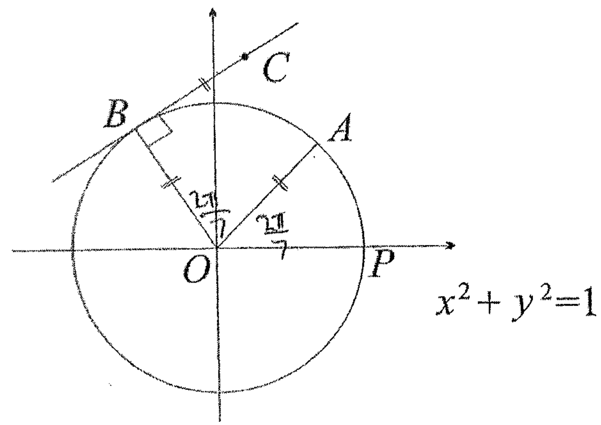
$= 1+2i$

$2+z$

Exam continues next page ...

QUESTION TWO (Continued)

(d)



In the diagram above, the points A and B lie on the circle $x^2+y^2=1$, so that

$$\angle AOP = \angle AOB = \frac{2\pi}{7}.$$

The point C lies on the tangent at B so that $CB = OB$, as shown.

(i) Write down, in modulus–argument form, the complex number represented by the point A. $\boxed{1}$

(ii) Write down, in modulus–argument form, the complex number represented by the point B. $\boxed{1}$

(iii) Let ω be the complex number represented by the point C. Explain why $\boxed{2}$

$$\text{Re}(\omega) = \cos\frac{4\pi}{7} + \sin\frac{4\pi}{7}.$$

$CB = i\,OB$

$\cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7} \cdot \text{cis}\,\frac{\pi}{2} \quad = \text{cis}\,\frac{\pi}{2} \cdot \text{cis}\,\frac{4\pi}{7}$

$CB = OB + AB$

$= \text{cis}\,\frac{2\pi}{7} + \text{cis}\,\frac{2\pi}{7}$

$= \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7} + \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}$

$= 2\cos\frac{4\pi}{7}$

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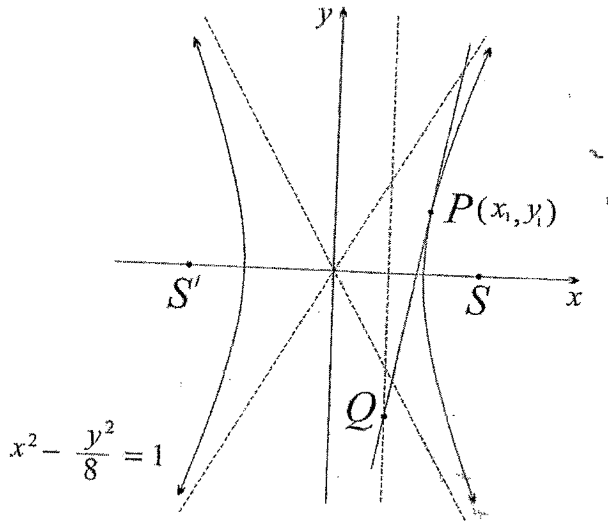
QUESTION THREE (15 marks) Use a separate writing booklet. Ma



$\frac{a - ae^2}{c}$

$= \frac{c}{a(1-e^2)}$

(a) The hyperbola $x^2 - \frac{y^2}{8} = 1$ is drawn above.

(i) Show that the eccentricity of the hyperbola is 3. [1]

(ii) Find the co-ordinates of the foci S and S' and the equations of the directrices. [2]

(iii) The tangent at the point $P(x_1, y_1)$ on the right-hand branch of the curve meets the nearer directrix at Q.

(α) Find the equation of the tangent at P. [2]

(β) Show that $\angle QSP = 90°$. -1 [3]

(b) The equation $x^3 - 3x^2 + 3x + \ell = 0$, where ℓ is rational, has $2 + i\sqrt{3}$ as a root.

(i) Find the other roots of the equation. Give reasons. [2]

(ii) What is the value of ℓ? [1]

(c) The equation $x^3 - x^2 + 3 = 0$ has roots α, β and γ.

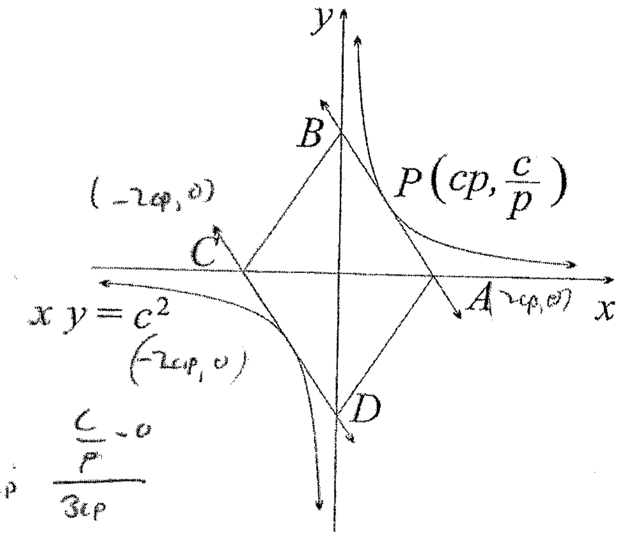
(i) Find the polynomial equation that has roots α^2, β^2 and γ^2. [2]

(ii) Find the value of $\alpha^4 + \beta^4 + \gamma^4$. [2]

QUESTION FOUR (15 marks) Use a separate writing booklet. Marl

(a) Solve the equation $z^2 = 2i$. [2]

(b)



$\frac{\frac{c}{p} - 0}{3cp}$

$= \frac{c}{p} \cdot 3cp$

$= 3c^2$

In the diagram above, the tangent at the point $P\left(cp, \frac{c}{p}\right)$ on the rectangular hyperbola $xy = c^2$ meets the x-axis at A and the y-axis at B. The parallel tangent meets the x-axis at C and the y-axis at D.

(i) Show that the equation of the tangent at P is $x + p^2y = 2cp$. [2]

(ii) Show that $ABCD$ is a rhombus. [2]

(iii) Show that the area of the rhombus $ABCD$ is $8c^2$. [2]

(iv) If the line CP is normal to the curve at P, prove that $3p^4 = 1$. [2]

(c) Let $P(z) = 3z - z^3 + 5z^4 + z^6$. You may assume that the six roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and α_6 are distinct.

(i) Prove that at least two of these roots are real. [2]

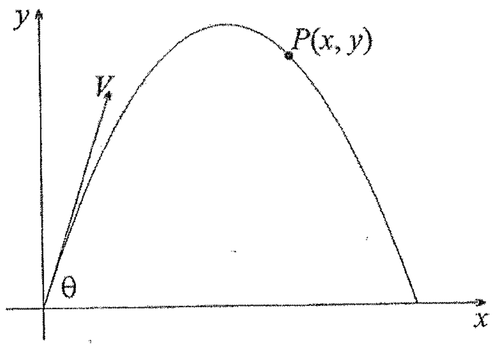
(ii) Explain why $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = 0$. [1]

(iii) Hence, or otherwise, show that there is at least one root with positive real part, and at least one root with negative real part. [2]

Exam continues overleaf ...

QUESTION FIVE (Continued)

(d) $\boxed{4}$



A particle is projected at an angle of θ to the horizontal with velocity V. (Assume that there is no air resistance.) At any time t, let x be the displacement in the horizontal direction and y be the displacement in the vertical direction.

You may assume that the motion satisfies the equations

$$\dot{x} = V\cos\theta \qquad \dot{y} = -gt + V\sin\theta$$
$$x = Vt\cos\theta \qquad y = -\tfrac{1}{2}gt^2 + Vt\sin\theta$$

(Do not derive these equations).

It is known that at some time t during its flight, the x and y displacements of the particle are equal and the direction of motion is inclined at 45° to the downward vertical. The position of the particle at this time is marked P in the diagram above.

Use this information to show that $\tan\theta = 3$.

QUESTION SIX (15 marks) Use a separate writing booklet.

(a) A ball of mass 1 kilogram is thrown vertically upwards from the ground with a speed of 10 metres per second. The ball is subject to a downward gravitational force and to air resistance of $\frac{v^2}{5}$ newtons, where v is the velocity of the ball. Take the value of g to be $10\,\text{m/s}^2$.

Thus the equation of motion of the ball, until it reaches its greatest height, is given by

$$\ddot{y} = -\frac{v^2}{5} - 10,$$

where y is the height in metres above the ground.

(i) Show that $y = -\frac{5}{2}\ln\left(\frac{v^2+50}{150}\right)$. $\boxed{3}$

(ii) Find the maximum height reached. $\boxed{1}$

(iii) Find the speed of the ball when it returns to the ground. $\frac{10\sqrt{3}}{3}$ $\boxed{3}$

(b) Let $I_n = \int_0^1 (1-x^2)^n\,dx$, where n is a positive integer.

(i) Show that $I_n = \frac{2n}{2n+1} I_{n-1}$. $\boxed{3}$

(ii) Find the value of I_3. $\boxed{2}$

(c) Show that the polynomial $\boxed{3}$

$$p(z) = z^n - z^{n-1} - 1, \text{ where } n \geq 2,$$

cannot have a repeated real root.

END OF EXAMINATION

$\left[\left[x-\frac{1}{3}x^3\right]\left[(1-x^2)^{n-1}\right]\right]' - 2(n-1)\int x(1-x^2)^{n-2}(x-\frac{1}{3}x^3)$

SGS Yr 12 Ext 2 2005. Half Yearly.

1.(a). $\int \frac{1}{x^2+4x+7}\,dx$

$\int \frac{dx}{x^2+4x+4+3}$ ✓

$\int \frac{dx}{(x+2)^2+3}$ ✓

$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+2}{\sqrt{3}}\right) + C.$

(b) (i). $\frac{8}{(x-2)(x^2-4x+8)} = \frac{A}{(x-2)} + \frac{Bx+C}{x^2+4x+8}$

$8 = A(x^2-4x+8) + (Bx+C)(x-2)$

Let $x=2$.

$8 = 4A$

$A = 2.$ ✓

Comparing x^2

$0 = A+B$

$0 = 2+B$

$B=-2.$ ✓

Comparing Constants.

$8 = 8A - 2C$

$8 = 16 - 2C$

$-8 = -2C$

$C = 4.$ ✓

$\frac{8}{(x-2)(x^2-4x+8)} = \frac{2}{(x-2)} + \frac{-2x+4}{(x^2-4x+8)}$

$\int \frac{8}{(x-2)(x^2-4x+8)}\,dx = \int \frac{2}{x-2}\,dx + \int \frac{-2x+4}{(x^2-4x+8)}\,dx.$ ✓

$= 2\int \frac{1}{x-2} - \int \frac{2x-4}{x^2-4x+8}\,dx$ ✓

$= 2\ln|x-2| - \ln|x^2-4x+8| + C$

(c). $\int_0^1 \frac{1}{(4-x^2)^{3/2}}\,dx$

Let $x = 2\sin\theta$. At $x=1$, $\theta = \pi/6$

$\frac{dx}{d\theta} = 2\cos\theta$ ✓ $x=0$, $\theta=0$

$dx = 2\cos\theta\,d\theta$

$\int_0^{\pi/6} \frac{2\cos\theta\,d\theta}{(4-4\sin^2\theta)^{3/2}}$

$\int_0^{\pi/6} \frac{2\cos\theta}{8\cos^3\theta}\,d\theta$

$\int_0^{\pi/6} \frac{1}{4}(\cos\theta)^{-2}$

$= \frac{1}{4}\int_0^{\pi/6} \cancel{(\cos\theta)^{-1}}$

$-1\times -\sin\theta$

$= \frac{1}{4}\left[\frac{\sin\theta}{\cos\theta}\right]_0^{\pi/6}$

-1

$= \frac{1}{4}\left[\frac{1}{\sin\theta\cos\theta}\right]_0^{\pi/6}.$

$= \frac{1}{4}\left[\frac{2}{\sqrt{3}}\times\frac{2}{\sqrt{3}}\right]$

$= \frac{1}{4}\int_0^{\frac{\pi}{6}} \frac{1}{\cos^2\theta}\,d\theta = \frac{1}{4}\int_0^{\pi/6} \sec^2\theta\,d\theta$

$= \frac{1}{4}\left[\tan\theta\right]_0^{\pi/6} = \frac{1}{4}\left[\frac{1}{\sqrt{3}}\right] = \frac{\sqrt{3}}{12}$

(d). $\int x\tan^{-1}x.$

$\int uv'\,dx = uv - \int vu'\,dx$

$u = \tan^{-1}x \qquad v = \frac{x^2}{2}.$ ✓

$u' = \frac{1}{1+x^2} \qquad v' = x$

$\int uv'\,dx = \frac{x^2}{2}\tan^{-1}x - \int \frac{x^2}{2(1+x^2)}\,dx$ ✓

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}$

$x^2+1 \overline{)\,x^2}$ $\quad 1$

-1

Let $I = \int \frac{x^2}{1+x^2}$ $\qquad \frac{x^2+1}{x^2} = 1 - \frac{1}{x^2+1}$

$= \int 1 - \int \frac{1}{1+x^2}\,dx$

$= x - \tan^{-1}x$ ✓

$\therefore \int x\tan^{-1}x = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left[x - \tan^{-1}x\right]$ ✓

$= \frac{1}{2}\left[x^2\tan^{-1}x - x + \tan^{-1}x\right]$

Question Two.

$z = 1-2i$

(i). $\bar{z} - iz$

$= (1+2i) - i(1-2i)$

$= 1+2i - i + 2$ ✓

$= -1+i$

(ii). $\frac{1}{2-z}$

$= \frac{1}{2-(1-2i)}$

$= \frac{1}{1+2i} \times \frac{1-2i}{1-2i}$

$= \frac{1-2i}{5}.$ ✓

$= \frac{1}{5} - \frac{2}{5}i$ ✓

(b)(i) $\sqrt{3} - i$

$= 2\operatorname{cis}\left(-\frac{\pi}{6}\right).$ ✓

(ii). $(\sqrt{3}-i)^8 = 2^7(a+ib)$

$= \left[2\operatorname{cis}\left(-\frac{\pi}{6}\right)\right]^8$

$= 2^8\operatorname{cis}\left(-\frac{8\pi}{6}\right)$ ✓

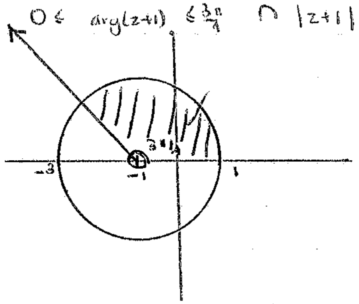
$= 2^8\operatorname{cis}\left(\frac{2\pi}{3}\right)$

$= 2^7\left[-1+\sqrt{3}i\right]$

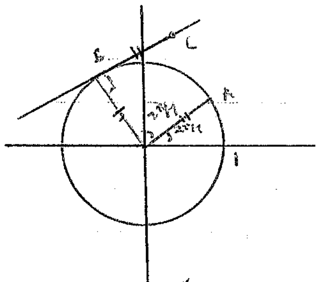
$a = -1,\ b = \sqrt{3}$ ✓

Question Two.

(c). $0 \leq \arg(z+1) \leq \frac{3\pi}{4}$ $\cap$ $|z+1| \leq 2$.



(d)



(i) $\underline{cis(2\pi/7)}$

* (ii). $\underline{cis(\frac{4\pi}{7})}$

$\therefore O = cis\frac{2\pi}{7}$

$CB = i\,OB$

$= i\ cis\frac{4\pi}{7}$

$CO = CB + BO$

$= iOB + OB$

$= \cancel{i\,OB}\ OB(i+1)$

$= cis\frac{4\pi}{7}(i+1)$

$= cis\frac{4\pi}{7} + i\,cis\frac{4\pi}{7}$

$w = cis\frac{4\pi}{7} + i\cos\frac{4\pi}{7} + \sin\frac{4\pi}{7}$

$\underline{Re(w) = \cos\frac{4\pi}{7} + \sin\frac{4\pi}{7}.}$

$\therefore \alpha^4+\beta^4+\gamma^4 = -8 - 3(1)$

$\underline{= -11.}$

Question Three

(b)(i) $x^3 - 3x^2 + 3x + l = 0$.

$2+i\sqrt{3}$ is root, $2-i\sqrt{3}$ is an ~~two other~~

(Conjugate root Theorem)

$\therefore$ ~~$x^2 - (\alpha+\beta)x + \alpha\beta$~~

$\therefore$ ~~$x^2 - 4x + 7$~~ is a factor.

$\alpha+\beta+\gamma = \frac{-b}{a}$

$4 + \gamma = 3.$

$\gamma = -1$

$\underline{\therefore -1 \text{ is another root.}}$

(ii). $\therefore P(-1) = 0$.

$-1 - 3 - 3 + l = 0.$

$\underline{l = 7.}$

(c)(i) $x^3 - x^2 + 3 = 0$.

$\alpha^2, \beta^2, \gamma^2$

Let ~~graph~~ $y = x^2$

~~$\sqrt{x}$ay~~ $x = \sqrt{y}$

$(\sqrt{y})^3 - y + 3 = 0.$

$y\sqrt{y} = y - 3$

$y^2 y = y^2 - 6y + 9.$

$y^3 - y^2 + 6y - 9 = 0.$

$\underline{\text{or } x^3 - x^2 + 6x - 9 = 0.}$

*(ii). $\alpha^4 + \beta^4 + \gamma^4$

If α is a root $\alpha^3 - \alpha^2 + 3 = 0$, times by α

$\alpha^4 - \alpha^3 + 3\alpha = 0$.

$\alpha^4 = \alpha^3 - 3\alpha$

Similarly $\beta^4 = \beta^3 - 3\beta$

$\gamma^4 = \gamma^3 - 3\gamma$

$\therefore \alpha^4 + \beta^4 + \gamma^4 = \alpha^3 + \beta^3 + \gamma^3 - 3(\alpha+\beta+\gamma$

$\Leftarrow$ $\alpha^3 = \alpha^2 - 3,\ \beta^3 = \beta^2 - 3,\ \gamma^3 = \gamma^2 - 3.$

$\therefore \alpha^3 + \beta^3 + \gamma^3 = \cancel{\alpha^2\beta^2}\ \alpha^2 + \beta^2 + \gamma^2 - 9$ (

$= (\alpha+\beta+\gamma)^2 - 2(\Sigma\alpha\beta) - 9.$

$= 1 - 9$

Question Four.

(a). $z^2 = 2i$

$2i = (a+ib)^2$

$2i = a^2 - b^2 + 2iab$

$a^2 - b^2 = 0$ $\qquad$ $2 = ab$

$\frac{2}{b} = a.$

$\frac{4}{b^2} - b^2 = 0.$

$4 - b^4 = 0.$

$b^4 = 4.$

$b = \pm\sqrt{2}.$

$\therefore a = \pm\sqrt{2}.$

$\therefore z = \pm(\sqrt{2} + \sqrt{2}i)$

$\underline{= \pm\sqrt{2}(1+i).}$

(c)(i). $P(z) = 3z - z^3 + 5z^4 + z^6$

$= z(z^5 + 5z^3 - z^2 + 3) = 0.$

$\therefore z = 0$ or $z^5 + 5z^3 - z^2 + 3 = 0.$

$\therefore 0$ is a real root. $\qquad$ Since $z^5 + 5z^3 - z^2 + 3$ is to the odd power, there is at least 1 real root.

(ii). $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = \frac{-b}{a}$ (Sum of Roots).

$= 0.$

(iii). Let two conjugate pairs be 2 sets of complex roots, Let α be the other real root

ie Roots $0,\ k,\ a+ib,\ a-ib,\ -c+id,\ -c-id.$

$\Sigma\alpha = -\frac{b}{a} = 0.$

$k + 2a + 2c = 0.$

$\Sigma\alpha\beta\gamma\delta\varepsilon = -\frac{f}{a} = -3$ $\qquad$ *Note:* ${}^6C_5 = 6$ *ways of which 5 products contain 0*

$\alpha\bar{\alpha}\beta\bar{\beta}.k = -3$

$\therefore (a+ib)(a-ib)(c+id)(c-id)k = -3$

$\therefore$ ~~mean~~ $\underset{+ve}{(a^2+b^2)}\underset{+ve}{(c^2+d^2)}(k) = -3$

and $\Sigma\alpha = 2a + 2c + k = 0$ $\qquad$ $\therefore k < 0$ at least root with negative real part.

... is positive.