SYDNEY GIRLS HIGH SCHOOL

2004 HSC Assessment Task 3

June 7th 2004

MATHEMATICS Extension 2

Year 12

Time allowed: 90 minutes

Topics: Polynomials, Integration, Volumes by Slicing (1st part)

DIRECTIONS TO CANDIDATES:

- Attempt all questions
- Questions are not of equal value
- There are 12 questions
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work
- Board approved calculators may be used
- Write on one side of the paper only

1. Find the following integrals

a) $\int \frac{dx}{\sqrt{16 - 9x^2}}$ b) $\int \frac{x^2.dx}{1 + x^6}$

c) $\int \frac{dx}{1 + \sin x}$ d) $\int \cos 2x.\sin x.dx$ [18]

e) $\int \frac{1 - 4x}{2x + 1}.dx$ f) $\int \frac{2dx}{(x+1)(x^2+1)}$

2. Evaluate the following integrals

a) $\int_0^2 \frac{dx}{\sqrt{4x - x^2}}$ b) $\int_{-2}^{2} \frac{x^2 + 2x^3 + \sin x}{x^2}.dx$

[12]

c) $\int_1^e \frac{\log_e x}{x^2}.dx$

3. a) Show that $\int \sin^n x.dx = -\frac{\cos x.\sin^{n-1} x}{n} + \frac{n-1}{n} \int \sin^{n-2} x.dx$ [8]

b) Using the result from part a) or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \sin^7 x.dx$

4. a) Show that $\int_0^a f(x).dx = \int_0^a f(a - x).dx$ and [8]

b) Hence or otherwise evaluate $\int_0^2 x\sqrt{2-x}.dx$

5. a) If a polynomial P(x) has a root of multiplicity m at x = a, [8]
show that $P'(x)$ has a root of multiplicity m-1 at x = a.
b) Find the roots of $4x^4 - 21x^3 + 30x^2 + 4x - 24 = 0$ given it has a three fold root

6. Find the roots of $4x^4 - 17x^3 - 4x^2 + 82x - 20 = 0$ given $x = 3 + i$ is a root. [5]

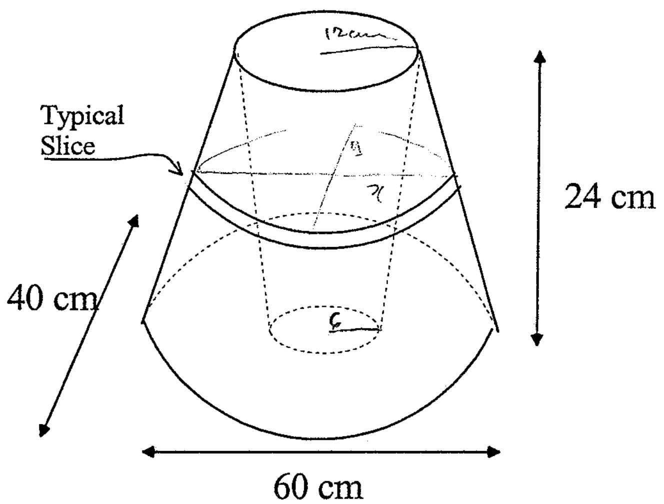
7. Factorise $P(x) = 48x^3 + 64x^2 - 27x - 36 = 0$ given that the sum of two of its roots is zero. [5]

8. If the roots of $2x^3 + 6x^2 - 4x - 1 = 0$ are α, β and γ; find
a) $\alpha^2 + \beta^2 + \gamma^2$
b) $\alpha^3 + \beta^3 + \gamma^3$
c) the polynomial equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ [8]
d) the polynomial equation with roots $\alpha^2, \beta^2, \gamma^2$

9. If each cross section perpendicular to the major axis of the ellipse $\frac{x^2}{9} + \frac{y^2}{1} = 1$ is a right angled isosceles triangle with the hypotenuse on the base,
find the volume of the solid. [5]

10. The base of a solid is the area enclosed by the curves $x^2 = 2y$ and $y = 3x - x^2$.
If each cross section perpendicular to the X axis is a semi circle,
find the volume of the solid. [7]

11. A vulcanologist wants to make a model of a volcano. The base of the model is elliptical in shape with the axes 60cm by 40 cm reducing uniformly to a circle of radius 12cm at the top. The hollow core of the model is a circle of radius 6cm at the base rising uniformly to a circle also of radius 12cm at the top. If the model is to be 24cm high find the volume of the material needed to construct the model. (The area of an ellipse with semi major axis A and semi minor axis B is πAB) [8]



12. a) Use de Moivre's theorem to express $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$ and hence find $\cot 4\theta$ in terms of $\cot\theta$.

b) Explain why $x = \cot\theta$ is a solution to $x^4 - 6x^2 + 1 = 0$ [8]

b) Find the value of $\cot^2\frac{\pi}{8} + \cot^2\frac{3\pi}{8}$.

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Math 4U - Term 2, 2004

1 a) $\int \frac{dx}{\sqrt{16-9x^2}} = \frac{1}{3}\sin^{-1}\left(\frac{3x}{4}\right) + c$

b) $\int \frac{x^2\,dx}{1+x^6}$ let $u = x^3$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$\therefore I = \frac{1}{3}\int \frac{du}{1+u^2} = \frac{1}{3}\tan^{-1}u + c$

$= \frac{1}{3}\tan^{-1}(x^3) + c$

(c) $\int \frac{dx}{1+\sin x}$. let $t = \tan x/2$

$\therefore \frac{2dt}{1+t^2} = dx$

$\frac{2t}{1+t^2} = \sin x$

$I = \int \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}}$

$= \int \frac{2dt}{1+t^2+2t}$

$= \int \frac{2dt}{(t+1)^2}$

$= -\frac{2}{t+1} + c$

$= \frac{-2}{\tan x/2 + 1} + c$

d) $\int \cos 2x \sin x\, dx$

$= \int (\cos^2 x - \sin^2 x)\sin x\, dx$

$= \int (2\cos^2 x - 1)\sin x\, dx$

Let $u = \cos x$, $du = -\sin x\, dx$

$\therefore I = -\int (2u^2-1)\,du$

$= \int (1-2u^2)\,du$

$= u - 2/3\,u^3 + c$

$= \cos x - \frac{2}{3}\cos^3 x + c$

e) $\int \frac{-4x+1}{2x+1}\,dx$

$= \int \left(\frac{-4x-2}{2x+1} + \frac{3}{2x+1}\right)dx$

$= -2x + \frac{3}{2}\ln(2x+1) + c$

f) $\int \frac{2\,dx}{(x+1)(x^2+1)}$

Let $\frac{2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

$\therefore 2 = A(x^2+1) + (Bx+C)(x+1)$

at $x = -1$ $2 = 2A$, $A = 1$

Equate x^2 $A+B=0$, $B = -1$

at $x = 0$ $2 = A+C$, $C = 1$

$\therefore I = \int \frac{dx}{x+1} - \int \frac{x\,dx}{x^2+1} + \int \frac{dx}{x^2+1}$

$= \ln(x+1) - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x + c$

(2) a) $\int_0^2 \frac{dx}{\sqrt{4x-x^2}}$

$-x^2 + 4x = -(x^2-4x+4)+4$

$= 4-(x-2)^2$

$= \int_0^2 \frac{dx}{\sqrt{4-(x-2)^2}}$

$= \left[\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^2$

$= \sin^{-1}(0) - \sin^{-1}(-1) = \pi/2$

b) $\int_{-2}^{2} \frac{x^2+2x^3+\sin x}{x^2}\,dx$

$= \int_{-2}^{2}\frac{x^2}{x^2}\,dx + \int_{-2}^{2}\frac{2x^3+\sin x}{x^2}\,dx$

$= 2\int_0^2 1\,dx$ (even) $+ 0$ (odd)

$= 2x\,]_0^2$

$= 4$

(2c, 3a at the end)

(12)

(a) $(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$

$= \cos 4\theta + i\sin 4\theta$ (de Moivre's thm)

Equating Real & Imaginary parts

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

$\therefore \cot 4\theta = \dfrac{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}$

$= \dfrac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$

If $\cot 4\theta = 0$, & $\cot^4\theta - 6\cot^2\theta + 1 = 0$

$4\theta = \pi/2, 3\pi/2, 5\pi/2, 7\pi/2$

& $\theta = \pi/8, 3\pi/8, 5\pi/8, 7\pi/8$

The equation $x^4 - 6x^2 + 1 = 0$, has, as its solutions $\cot \pi/8$, $\cot 3\pi/8$, $\cot 5\pi/8$, $\cot 7\pi/8$

& $\cot\pi/8 . \cot 3\pi/8 + \cot\pi/8 . \cot 5\pi/8 + \cot\pi/8 . \cot 7\pi/8 + \cot 3\pi/8 . \cot 5\pi/8$

$+ \cot 3\pi/8 . \cot 7\pi/8 + \cot 5\pi/8 \cot 7\pi/8 = \frac{c}{a} = -6$

But $\cot 5\pi/8 = -\cot 3\pi/8$, $\cot 7\pi/8 = -\cot \pi/8$

$\therefore \cot\frac{\pi}{8}\cot 3\pi/8 - \cot\pi/8\cot 3\pi/8 - \cot\frac{\pi}{8}.\cot\pi/8 - \cot 3\pi/8 . \cot 3\pi/8$

$+ \cot 3\pi/8(-\cot\pi/8) + \cot\frac{3\pi}{8}\cot\pi/8 = -6$

$\therefore -\cot^2\pi/8 - \cot^2 3\pi/8 = -6$

& $\cot^2\pi/8 + \cot^2 3\pi/8 = 6.$

(2)(c)

$\int_1^e \frac{\ln x}{x^2}\,dx$

$u = \ln x \qquad dv = \frac{1}{x^2}dx$

$du = \frac{1}{x}dx \qquad v = -\frac{1}{x}$

$\therefore I = \left[-\frac{\ln x}{x} + \int \frac{1}{x^2}dx\right]_1^e$

$= \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^e$

$= \left[\frac{\ln x}{x} + \frac{1}{x}\right]_e^1$

$= 0 + 1 - \frac{1}{e} - \frac{1}{e}$

$= 1 - 2/e.$

3. a $\int \sin^n x.dx = \int \sin^{n-1}x.\sin x.dx$

$u = \sin^{n-1}x$

$du = (n-1)\cos x.\sin^{n-2}x.dx$

$dv = \sin x.dx$

$v = -\cos x$

$\therefore I = -\cos x.\sin^{n-1}x + n-1\int \cos^2 x\sin^{n-2}$

$= -\cos x\sin^{n-1}x + (n-1)\int (1-\sin^2 x)\sin^{n-2}x.d$

$= -\cos x.\sin^{n-1}x + (n-1)\int \sin^{n-2}x.dx - (n-1)\int \sin^n x.d$

$\therefore n.I_n = -\cos x\sin^{n-1}x + (n-1)I_{n-2}$

$$\therefore \int_0^{\pi/2} \sin^7 x\,dx = \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\int_0^{\pi/2} \sin x\,dx$$

$$= \frac{8}{35}\left[-\cos x\right]_0^{\pi/2}$$

$$= 8/35$$

4a) $\int_0^a f(x)\,dx.$

let $x = a - u$ | $x = 0,\ u = a$

$\therefore dx = -du$ | $x = a,\ u = 0$

$$\therefore \int_0^a f(x)\,dx = -\int_a^0 f(a-u)\,du$$

$$= \int_0^a f(a-u)\,du$$

$$= \int_0^a f(a-x)\,dx.$$

b) $\int_0^2 x\sqrt{2-x}\,dx$

$$= \int_0^2 (2-x)\sqrt{2-(2-x)}\,dx$$

$$= \int_0^2 (2-x)\sqrt{x}\,dx$$

$$= \int_0^2 2x^{1/2} - x^{3/2}\,dx$$

$$= \frac{4}{3}x^{3/2} - \frac{2}{5}x^{5/2}\Big]_0^2$$

$$= \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5} = \frac{16\sqrt{2}}{15}$$

5a) let $P(x) = (x-a)^m Q(x)$

$$\therefore P'(x) = m(x-a)^{m-1}Q(x) + (x-a)^m Q'(x)$$

$$= (x-a)^{m-1}\{mQ(x) + (x-a)Q'(x)\}$$

$$= (x-a)^{m-1}Q^*(x),$$

a polynomial with a root of multiplicity $m-1$ at $x = a$.

b) $P(x) = 4x^4 - 21x^3 + 30x^2 + 4x - 24$

$$P'(x) = 16x^3 - 63x^2 + 60x + 4$$

$$P''(x) = 48x^2 - 126x + 60$$

$$= 6(8x^2 - 21x + 10)$$

$$= 6\,\frac{(8x-16)(8x-5)}{8}$$

$$= 6(x-2)(8x-5)$$

$P(5/8) \neq 0$, $P(2) = 0$

$$\therefore P(x) = (x-2)^3(4x+3)$$

(6) If $3+i$ is a root, so is $3-i$

$$\therefore P(x) = (x-3-i)(x-3+i)(ax^2+bx+c)$$

$$= (x^2-6x+10)(ax^2+bx+c)$$

$$= 4x^4 - 17x^3 - 4x^2 + 82x - 20$$

$\therefore a = 4,\ c = -2,\ b = 7$

$$\therefore P(x) = (x^2-6x+10)(4x^2+7x-2)$$

$$\therefore P(x) = (x-3-i)(x-3+i)(4x-1)(x+2)$$

$\therefore$ Roots are $3+i,\ 3-i,\ \frac{1}{4},\ -2$

(7) let roots be $a, -a, b$

$$P(a) = 48a^3 + 64a^2 - 27a - 36 = 0$$

$$P(-a) = -48a^3 + 64a^2 + 27a - 36 = 0$$

ing $128a^2 - 72 = 0$

$$\therefore a^2 = \frac{72}{128} = \frac{36}{64} = \frac{9}{16}$$

$\therefore a = 3/4,\ -a = -3/4$

$$\& P(x) = (4x+3)(4x-3)(3x+4)$$

(8). $2x^3 + 6x^2 - 4x - 1 = 0$ has roots α, β, γ

$\Sigma\alpha = -3$, $\Sigma\alpha\beta = -2$

$$\therefore \Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$$

$$= 9 + 4 = 13$$

$f(\alpha) = f(\beta) = f(\gamma) = 0$

$$\therefore 2\alpha^3 + 6\alpha^2 - 4\alpha - 1 = 0$$

$$2\beta^3 + 6\beta^2 - 4\beta - 1 = 0$$

$$2\gamma^3 + 6\gamma^2 - 4\gamma - 1 = 0$$

$$\therefore 2\Sigma\alpha^3 = -6\Sigma\alpha^2 + 4\Sigma\alpha + 3$$

$$\therefore \Sigma\alpha^3 = (-78 - 12 + 3) \div 2$$

(8)

same as $(\frac{1}{x}-\alpha)(\frac{1}{x}-\beta)(\frac{1}{x}-\gamma)=0$

∴ Replace x by $1/x$ in original eqn

$\therefore \frac{2}{x^3}+\frac{6}{x^2}-\frac{4}{x}-1=0$

$\therefore 2+6x-4x^2-x^3=0$

OR $x^3+4x^2-6x-2=0$.

d) Replace x by $\sqrt{x}$

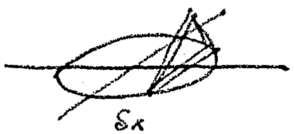
$\therefore 2x\sqrt{x}+6x-4\sqrt{x}-1=0$

$\therefore \sqrt{x}(2x-4)=1-6x$

$\therefore x(4x^2-16x+16)=1-12x+36x^2$

$\therefore 4x^3-52x^2+28x-1=0$

9.



$V_{SLICE} = \frac{1}{2}bh\,\delta x$

$= \frac{1}{2}\cdot 2y\cdot y\,\delta x$

$= y^2 dx = (1-\frac{x^2}{9})\,\delta x$

$\therefore V_{SOLID} = \sum_{-3}^{3}(1-\frac{x^2}{9})\,\delta x$

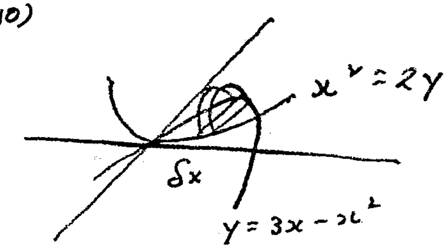
$= 2\int_0^3 (1-\frac{x^2}{9})\,dx$

$= 2\left[x-\frac{x^3}{27}\right]_0^3$

$= 2(3-1)$

∴ Volume = 4 units^3

(10)



Equating for y: $\frac{x^2}{2}=3x-x^2$

$x^2=6x-2x^2$

$0=6x-3x^2$

$0=3x(2-x)$

$\therefore x=0,2$

$V_{SLICE}=\frac{1}{2}\pi r^2\,\delta x$

$r=\frac{y_1-y_2}{2}=\frac{3x-x^2-x^2/2}{2}$

$=\frac{3x-3x^2/2}{2}$

$=\frac{6x-3x^2}{4}=\frac{3x(2-x)}{4}$

$\therefore r^2=\frac{9x^2}{16}(4-4x+x^2)$

$\therefore V_{SOLID}=\sum_{x=0}^{2}\frac{\pi}{2}\cdot\frac{9}{16}(4x^3-4x^2+x^4)\,\delta x$

$=\int_0^2 \frac{9\pi}{32}(4x^2-4x^3+x^4)\,dx$

$=\frac{9\pi}{32}\left[\frac{4x^3}{3}-x^4+\frac{x^5}{5}\right]_0^2$

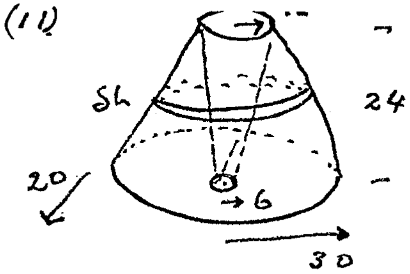
$=\frac{9\pi}{32}\left[\frac{32}{3}-16+\frac{32}{5}\right]$

$=\pi\left[3-\frac{9}{2}+\frac{9}{5}\right]$

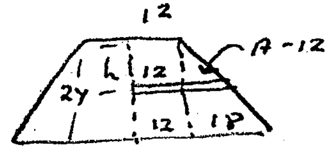
$=\pi\left[\frac{30-45+18}{10}\right]=\frac{3\pi}{10}$

∴ Volume $=\frac{3\pi}{10}$ u^3.

(11)



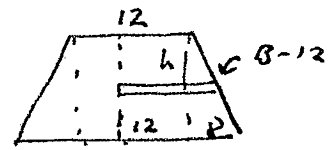
$V_{SLICE}=(\pi AB-\pi R^2)\,\delta h$.



By similar Δ's

$\frac{A-12}{18}=\frac{h}{24}$

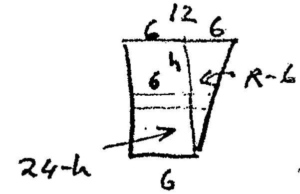
$\therefore A=12+\frac{3h}{4}$



By similar Δ's

$\frac{B-12}{8}=\frac{h}{24}$

$B=12+\frac{h}{3}$



By similar Δ's

$\frac{R-6}{6}=\frac{24-h}{24}$

$R-6=\frac{24-h}{4}$

$\therefore R=12-\frac{h}{4}$

$\therefore V_{SLICE}=\pi\left\{(12+\frac{3h}{4})(12+\frac{h}{3})-(12-\frac{h}{4})^2\right\}\delta h$

$=\pi\left\{144+13h+\frac{h^2}{4}-144+6h-\frac{h^2}{16}\right\}\delta h$

$=\pi\left\{\frac{3h^2}{16}+19h\right\}\delta h$

$\therefore V_{SOLID}=\pi\int_0^{24}(\frac{3h^2}{16}+19h)\,dh$

$=\pi\left[\frac{h^3}{16}+\frac{19h^2}{2}\right]_0^{24}$

∴ Volume = 6336π cm^3