SYDNEY GIRLS HIGH SCHOOL

2000

MATHEMATICS

3 UNIT

June Assessment

Time allowed - 75 minutes

DIRECTIONS TO CANDIDATES NAME [redacted]

- Attempt ALL questions.
- Questions are not of equal value - marks are shown
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Each question attempted should be started on a new sheet. Write on one side of the paper only.

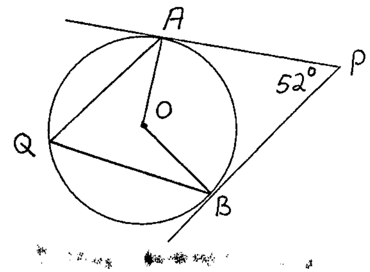
Question One (17 Marks)

i) Evaluate the following integrals:

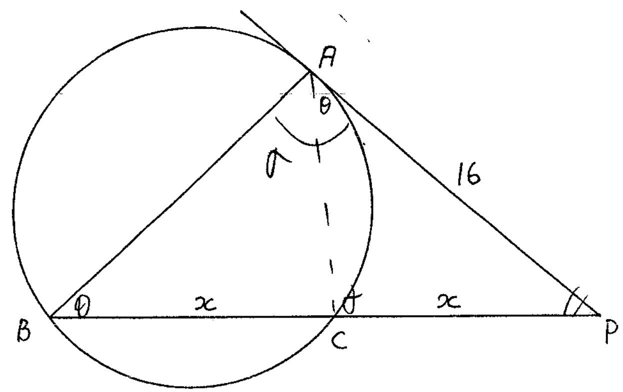
a) $\int_0^{\frac{\pi}{4}} (2 - \tan^2 x)dx$

b) $\int_0^{\frac{\pi}{4}} (2\cos^2 x - \cos x)dx$

ii) AP and BP are tangents.
O is the centre of the circle
Copy this diagram onto your examination page
Find <AQB, giving reasons



iii) Given BC = PC = x,
Find x, giving reasons
(leave in exact form)

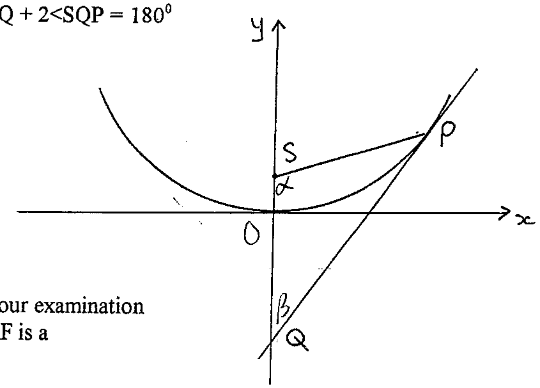


Question Two (16 Marks)

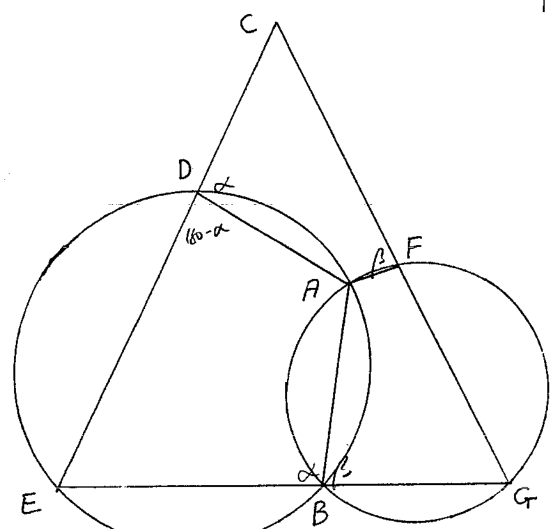
i) a) Differentiate $\sin x - x\cos x$ with respect to x.
b) Hence or otherwise evaluate

$$\int_0^{\frac{\pi}{2}} x \sin x \, dx$$

ii) The point $P(2ap,\ ap^2)$ lies on the parabola $x^2 = 4ay$.
The focus S is the point (0,a).
The tangent at P meets the Y axis at Q
a) Find the co-ordinates of Q
b) Prove SP=SQ
c) Hence show $\angle PSQ + 2\angle SQP = 180^0$



iii) Copy the figure onto your examination paper and prove CDAF is a cyclic quadrilateral

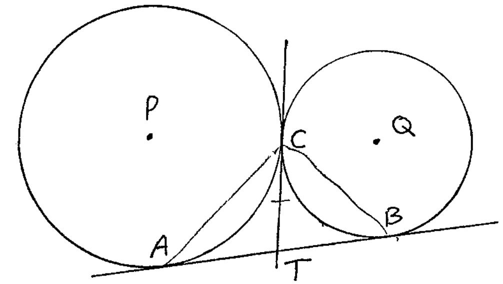


Question Three (13 Marks)

i) By letting $t = \tan\frac{\theta}{2}$ or otherwise, find all the values of θ in the range $0 \le \theta \le \pi$ such that $2\sin\theta + \cos\theta = 1$

$t^2 - 2t = 0$
$t(t-2) = 0$
$t = 0$ or $t = 2.$

ii) Two circles touch externally at C. The circles which have centers P and Q are touched by a common tangent at A and B respectively. The common tangent at C meets AB in T.



a) Copy this diagram onto your examination paper
b) Prove AT= TB
c) Show $\angle ACB = 90^0$

Question Four (18 Marks)

i) Solve the equation $2\sec^2\theta - 3\tan\theta = 1$ for $0 \le \theta \le 2\pi$
Give your answer in radians correct to one decimal place.

ii) Consider the parabola $y = x^2$ $a = \frac{1}{4}$ $2at,\ at^2$ $4a = 1$

a) Show the equation of the tangent *(l)* to the parabola at the point $a = \frac{1}{4}.$ $T(t, t^2)$ is given by $y = 2tx - t^2$
b) Show that the line passing through the focus of the parabola and perpendicular to *(l)* has the equation $y = \dfrac{t-2x}{4t}$
c) Hence or otherwise find the locus of the foot of the perpendicular from the focus to the tangent to the parabola at any point.

$\frac{3x^2}{2a^2}$

$\frac{\alpha}{2}$ ✓ $\frac{3x^2}{2a^2}$

$\frac{3x^2}{4a}$

$\therefore 4ay = 3x^2.$

$\therefore x = \frac{1}{4}t$

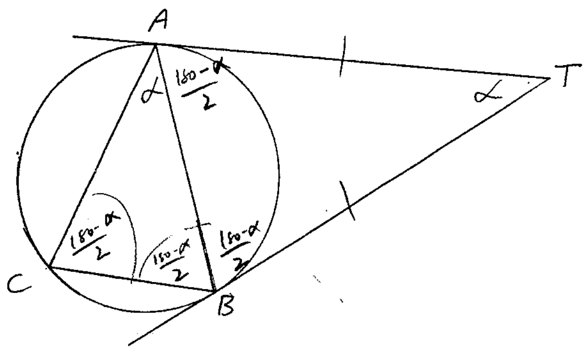
$y - t^2 = 2t(x - t)$
$= 2tx - 2t^2$

Question Five (18 Marks)

i) Prove the identity $\dfrac{\cos x - \cos(x+2y)}{2\sin y} = \sin(x+y)$

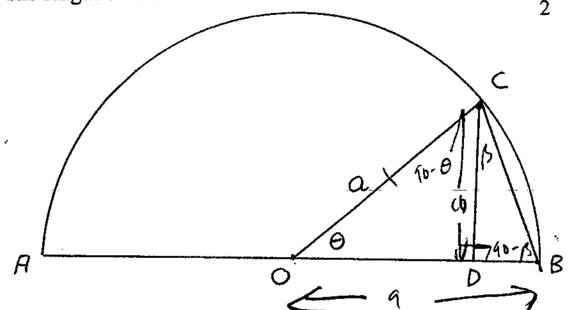
ii) By expressing $\sqrt{3}\cos x - \sin x$ in the form $A\cos(x+\alpha)$, solve the equation $\sqrt{3}\cos x - \sin x = 1$ for $0 \le x \le 2\pi$

iii) TA, TB are tangents
<BAC = <ATB
Copy this diagram onto your Examination paper
Prove AC = AB



Question Six (18 Marks)

i) The length of OC is a units. Show that BD is: $a\sin\theta\tan\dfrac{\theta}{2}$ $a\sin\theta \cdot \dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$



$a \times \dfrac{2t}{1+t^2}$, t

$\tan\beta = \dfrac{CD}{BD}$

iii) The tangents at P $(2ap, ap^2)$ and Q $(2aq, aq^2)$ on the parabola $x^2 = 4ay$ meet at R on the parabola $x^2 = -4ay$. $\tan\dfrac{\theta}{2} = \dfrac{a\sin\theta}{BD}$

Show that the locus of the midpoint of the chord PQ is $3x^2 = 4ay$.
(You may write down the equations of the tangents at P and Q).

$\cancel{90} - \beta = \cancel{90} - \theta + \beta$

$\angle B$

Question Two

(i) a) $f(x) = \sin x - x.\cos x$

$f'(x) = \cos x - 1.\cos x - x.(-\sin x)$

$= \underline{x.\sin x.}$ (2)

b) $I = \int_0^{\frac{\pi}{2}} x.\sin x.\,dx = \left[\sin x - x.\cos x\right]_0^{\frac{\pi}{2}}$

$= \left[\sin\frac{\pi}{2} - 0\right] - \left[0 - 0\right]$

$= \underline{\underline{1}}$ (2)

(ii) a) $x^2 = 4ay \Rightarrow y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a}$ (at $P(2ap, ap^2)$)

$\frac{dy}{dx} = \frac{2ap.2}{4a} = p.$

Eqn of tangent at $P(2ap, ap^2)$.

$y - ap^2 = p(x - 2ap)$

$\underline{y = px - ap^2}$ *

Co-ords of Q on y-axis when $x = 0$ $y = \underline{-ap^2}$

$Q(\underline{0, -ap^2}).$ (3)

b) Distance $SQ = |a| + |-ap^2| = a + ap^2.$

Distance $SP = \sqrt{(2ap - 0)^2 + (ap^2 - a)^2}$

$= \sqrt{4a^2p^2 + a^2p^4 - 2a^2p^2 + a^2}$

$= a\sqrt{p^4 + 2p^2 + 1} = a\sqrt{(p^2+1)^2}$

$= a(p^2 + 1)$

$\therefore \underline{SQ = SP}$ (3)

c) In $\triangle PSQ$

$SQ = SP$ (i.e. isosceles $\triangle$)

$\therefore \angle SQP = \angle SPQ.$

Hence, angle sum of $\triangle PSQ$

$\Rightarrow \underline{\angle PSQ + 2\angle SQP = 180°}$ (2)

Question One

(i) a) $I = \int_0^{\frac{\pi}{4}} (2 - \tan^2 x).\,dx$

$= \int_0^{\frac{\pi}{4}} (2 - (\sec^2 x - 1)).\,dx$

$= \int_0^{\frac{\pi}{4}} (3 - \sec^2 x).\,dx.$

$= \left[3x - \tan x\right]_0^{\frac{\pi}{4}}$

$= \frac{3\pi}{4} - 1 - 0$

$= \underline{\underline{\frac{3\pi}{4} - 1}}$ (4)

b) $\int_0^{\frac{\pi}{4}} (2\cos^2 x - \cos x).\,dx$

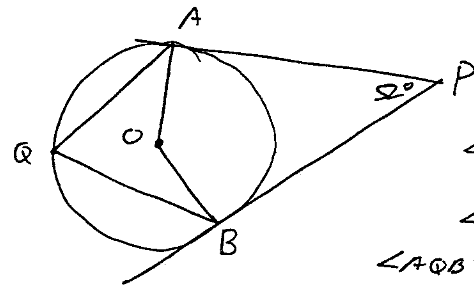
$= \int_0^{\frac{\pi}{4}} (\cos 2x + 1 - \cos x)\,dx$

$= \left[\frac{1}{2}\sin 2x + x - \sin x\right]_0^{\frac{\pi}{4}}$

$= \left[\frac{1}{2} + \frac{\pi}{4} - \frac{1}{\sqrt{2}}\right] - 0$

$= \underline{\frac{1}{2} + \frac{\pi}{4} - \frac{1}{\sqrt{2}}} =$

(ii)



$\angle PAO = \angle PBO = 90°$

(radii of a circle meets tangents at 90°

$\angle AOB = 360° - 90° - 90° - 52°$

($\angle$ sum of a quad.

$\angle AOB = 128°.$

$\angle AQB = 64°$ ($\frac{1}{2}\angle AOB$ - angle at centre is double angle at circumference on same arc

(5)

(iii) $AP^2 = PC.BP$

$16^2 = x.2x$

$16^2 = 2x^2$

$256 = 2x^2$

$128 = x^2$

$\underline{\underline{x = 8\sqrt{2}}}$ (4)

Q.3

i) $2\sin\theta + \cos\theta = 1$

Let $t = \tan\frac{\theta}{2}$, $0 \le \theta \le \pi$

Now $\sin\theta = \frac{2t}{1+t^2}$

$\cos\theta = \frac{1-t^2}{1+t^2}$

$\Rightarrow 2\left(\frac{2t}{1+t^2}\right) + \left(\frac{1-t^2}{1+t^2}\right) = 1$

$4t + (1-t^2) = (1+t^2).$

$4t + 1 - t^2 = 1 + t^2$

$4t = 2t^2$

$\therefore t = 0 \quad t = 2$

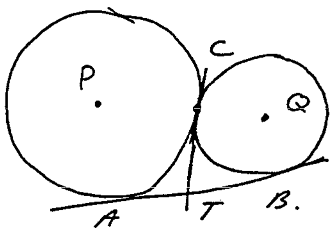
when $\tan\frac{\theta}{2} = 0$ $\quad \tan\frac{\theta}{2} = 2$

$\Rightarrow \underline{\theta = 0}$ $\quad \frac{\theta}{2} =$

$\theta = 2.21$ radians

7

(ii)



a) R.T.P. $AT = TB$

$AT = TC$ (tangents from ext pt to same circle)

$TB = TC$ (" ")

$\therefore \underline{AT = TB.}$

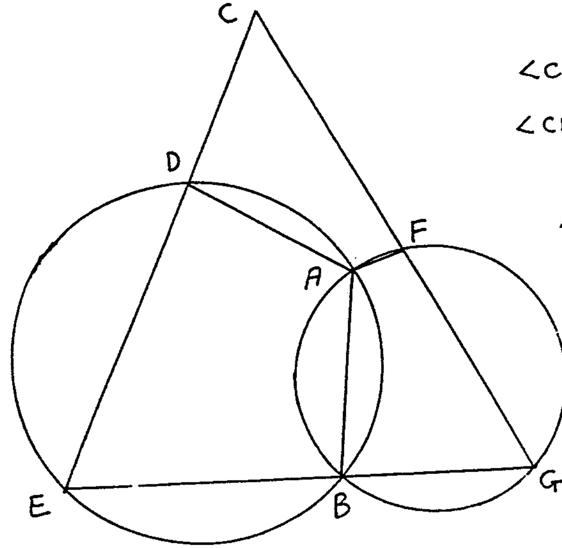
(b) As $AT = CT = BT$ then A, C and B all lie on a circle centre T with radius AT.

Hence $\angle ACB$ is on the circumference of this circle with diameter AB.

Hence, $\angle ACB = 90°$ (angle in semi-circle is a right angle)

6

iii) Copy the figure onto your examination paper and prove CDAF is a cyclic quadrilateral



$\angle CDA = \angle ABE$ (ext $\angle$ of cyclic quad ABED)

$\angle CFA = \angle ABG$ (ext $\angle$ of cyclic quad ABGF)

Now $\angle ABE + \angle ABG = 180°$ (straight angle)

$\therefore \angle CDA + \angle CFA = 180°$

$\therefore$ CDAF is a cyclic quad (one pair of opposite $\angle$s supplementary)

4

Q4 (i) $2\sec^2\theta - 3\tan\theta = 1.$

$2(1+\tan^2\theta) - 3\tan\theta = 1.$

$2 + 2\tan^2\theta - 3\tan\theta - 1 = 0.$ 1

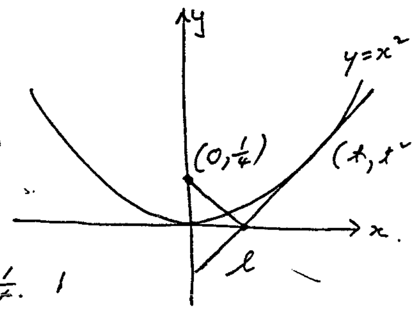
$2\tan^2\theta - 3\tan\theta + 1 = 0.$ 1

$(2\tan\theta - 1)(\tan\theta - 1) = 0$ 1

$\therefore \tan\theta = 1, \quad \tan\theta = \frac{1}{2}.$ 1 6

$\therefore \theta = \frac{\pi}{4}, \frac{5\pi}{4}, 0.46^c, 3.6^c$ 2

(ii) (a) $y = x^2 \qquad a = \frac{1}{4}.$



$\frac{dy}{dx} = 2x$ 1

at (t, t^2) $\frac{dy}{dx} = 2t.$ 1

Eqn of tangent

$y - t^2 = 2t(x - t)$ 1

$\boxed{y = 2tx - t^2}$ * 4

(b) Grad of perpendicular $\Rightarrow -\frac{1}{2t}.$ 1

Eqn. thru Focus $(0, \frac{1}{4})$ 1

$y - \frac{1}{4} = -\frac{1}{2t}(x - 0) \quad \times 4t.$

$4ty - t = -2(x)$ 1

$4ty = t - 2x.$ 1

$\boxed{y = \frac{t-2x}{4t}}$ * 4

(c) Locus of intersection of tangent and perpendicular

$y = 2tx - t^2, \quad y = \frac{t-2x}{4t}$

$2tx - t^2 = \frac{t-2x}{4t} \Rightarrow 8t^2x - 4t^3 = t - 2x$

$2tx$

$\Rightarrow 8t^2x + 2x = t + 4t^3$

$2x(4t^2+1) = t(1+4t^2) \Rightarrow 2x = t$

$x = \frac{t}{2}$ *

Now $y = 2tx - t^2$

$= 2t(\frac{t}{2}) - t^2 = 0 \Rightarrow y = 0$ 4

$\therefore$ Locus is x-axis

18

Q5

(i) $\frac{\cos x - \cos(x+2y)}{2\sin y} = \sin(x+y)$

L.H.S. $= \frac{\cos x - \{\cos x.\cos 2y - \sin x.\sin 2y\}}{2\sin y}$

$= \frac{\cos x - \{\cos x.(1 - 2\sin^2 y)\} - \sin x.2\sin y.\cos y\}}{2\sin y}$

$= \frac{\cancel{\cos x - \cos x} + \cos x.2\sin^2 y + \sin x.2\sin y\cos y}{2\sin y},$

$= \frac{2\sin y(\cos x.\sin y + \sin x.\cos y)}{2\sin y}$

$= \cos x\sin y + \sin x.\cos y$

$= \sin(x+y) = R.H.S.$

6

(ii) $\sqrt{3}\cos x - \sin x. \qquad A = \sqrt{(\sqrt{3})^2 + 1^2} = 2.$

Now $\sqrt{3}\cos x - \sin x = 2(\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x).$

Let $\cos\alpha = \frac{\sqrt{3}}{2} \quad \sin\alpha = \frac{1}{2}$

$\sqrt{3}\cos x - \sin x = 2\cos(x + \frac{\pi}{6}) \qquad \therefore \alpha = \frac{\pi}{6}.$

Now $\sqrt{3}\cos x - \sin x = 1 \Rightarrow 2\cos(x + \frac{\pi}{6}) = 1$

$\cos(x + \frac{\pi}{6}) = \frac{1}{2}.$ 1

$\cos\frac{\pi}{3} = \frac{1}{2} \qquad \cos\frac{5\pi}{3} = \frac{1}{2}.$

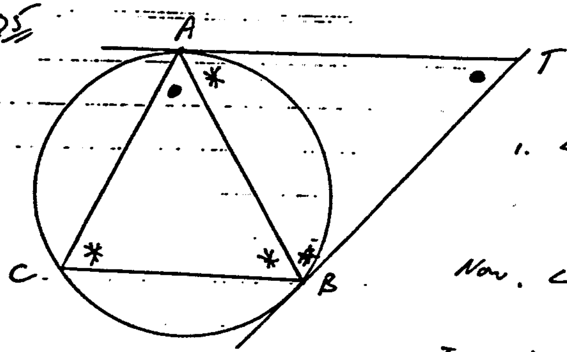
$\therefore x + \frac{\pi}{6} = \frac{\pi}{3} \qquad x + \frac{\pi}{6} = \frac{5\pi}{3}$

* $x = \frac{\pi}{6} \qquad \therefore x = \frac{9\pi}{6} = \frac{3\pi}{2}$ *

$x = \frac{\pi}{6}, \frac{3\pi}{2}$

6

Q5



1. $\angle TAB = \angle TBA$ (base $\angle$s of isos Δ
$TA = TB$
(equal tangents)

Now, $\angle TAB = \angle ACB$ ($\angle$ in alt. seg)

In ΔTAB $\angle ATB + 2\times\angle TAB = 180°$.

In ΔABC $\angle BAC = \angle ATB$

and $\angle ABC = \angle TAB$.

$\therefore$ $\angle ABC = \angle TAB$.

i.e. $\angle ACB = \angle ABC$

Hence $AB = AC$ (base $\angle$s of isos Δ equal)

6 ii) $\frac{x}{a} = (p+q)$

$$\frac{2y}{a} = p^2+q^2$$
$$= (p+q)^2 - 2pq$$
$$= (p+q)^2 - \frac{2(p+q)^2}{-4} \quad \text{from (1)}$$
$$\frac{2y}{a} = \frac{3}{2}(p+q)^2 \implies \frac{2y}{a} = \frac{3}{2}\frac{x^2}{a^2}$$
$$\therefore 4ay = 3x^2$$

(Gr2)

Q6 ii) $M\left(a(p+q), \frac{a}{2}(p^2+q^2)\right)$

ie $\frac{x}{a} = p+q$ $\quad \frac{2y}{a} = p^2+q^2$

$$\frac{x^2}{a^2} = (p+q)^2$$

$(p+q)^2 = -4pq$ (from (1))

$(p+q)^2 = p^2+q^2+2pq$

$\therefore p^2+q^2+2pq = -4pq$

$p^2+q^2 = -6pq$

$\therefore \frac{2y}{a} = -6pq$ — (2)

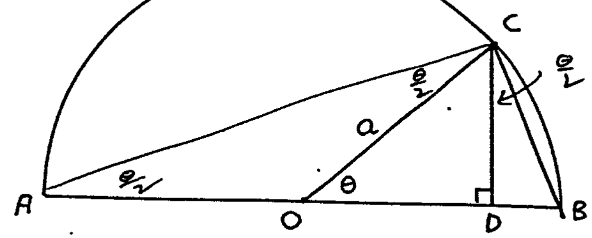
and $\frac{x^2}{a^2} = -4pq$ — (3)

$\therefore \frac{2y}{-6a} = \frac{x^2}{-4a^2}$

$-6ax^2 = -8a^2y$

$3x^2 = 4ay$

i) The length of OC is a units. Show that BD is: $a\sin\theta.\tan\frac{\theta}{2}$



Proof: Join AC.

Now ΔAOC is isosceles base angles $\frac{\theta}{2}$.

Now $\angle OCD = 90 - \theta$

But $ACB = 90°$ ($\angle$ in semi circle)

$\therefore \angle ACO + \angle OCD + \angle DCB = 90°$

$\Rightarrow \frac{\theta}{2} + 90 - \theta + \angle DCB = 90$

$\therefore \angle DCB = \frac{\theta}{2}$

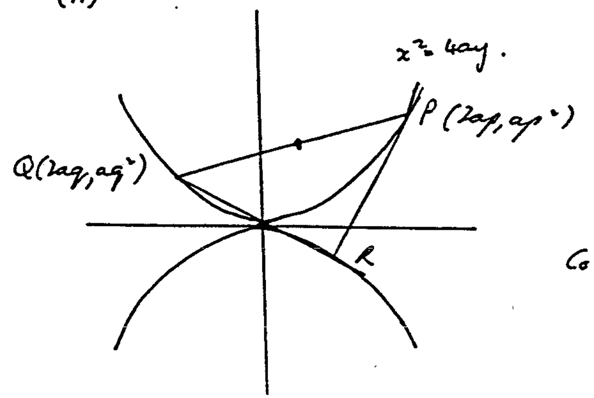
In ΔOCD $\sin\theta = \frac{CD}{a}$

$CD = a\sin\theta$

In ΔCDB $\tan\frac{\theta}{2} = \frac{DB}{CD}$.

$\therefore DB = a.\sin\theta.\tan\frac{\theta}{2}$

(ii)



Eqn of tangent at P.

$y = px - ap^2$

Eqn of tangent at Q

$y = qx - aq^2$.

Co-ords of R.

$px - ap^2 = qx - aq^2$.

$x(p-q) = ap^2 - aq^2$

$x = a(p+q)$

$y = ap(p+q) - ap^2 = apq$

Co-ords of R $(a(p+q), apq)$

But R lies on $x^2 = -4ay$.

$\therefore a^2(p+q)^2 = -4.a.apq$.

$\therefore (p+q)^2 = -4pq$. (1) $\Rightarrow (p+q)^2 + 4pq = C$.

Now mid-point $x = \frac{2ap+2aq}{2} = a(p+q)$ $\quad y = \frac{ap^2+aq^2}{2}$

$= \frac{a}{2}(p^2+q^2)$

Satisfies $3x^2 = 4ay$

$3a^2(p+q)^2 = 4a.\frac{a}{2}(p^2+q^2)$

PTO

$3(p+q)^2 = 2(p^2+q^2)$

$3(p^2+q^2+2pq) = 2p^2+2q^2$.

$3p^2+3q^2+6pq = 2p^2+2q^2$

$(p+q)^2+4pq$ shown above $\therefore$ lies on $x^2=-4$