YR 12 3U TERM 1 1999

75 MINUTES

• polynomials
• trig fns

Question 1

3 (a) Sketch and label showing relevant points

(i) $y = 2^x$

(ii) $y = e^{-x} - 1$

(iii) $y = \log_e(x-1)$

3 (b) Differentiate

(i) $y = e^{3x^2}$

(ii) $y = \dfrac{\log_e x}{x}$

3 (c) Find (i) $\int e^{ax+b}\,dx$

(ii) $\int xe^{-x^2}\,dx$

(iii) $\int \dfrac{6}{3x+5}\,dx$

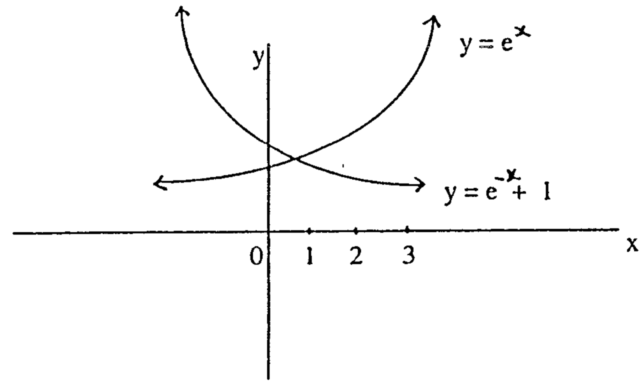
3 (d) Solve $3\log_4 2 = \log_4 x - \log_4 6$

Question 2

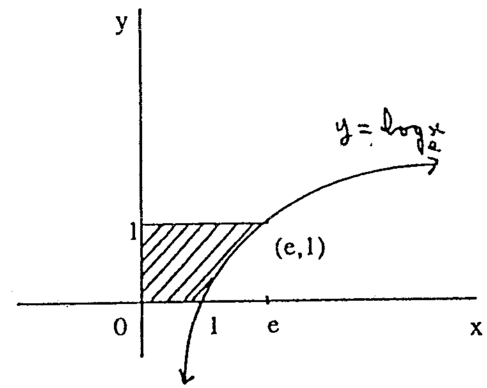
3 (a) Find the equation of the tangent to the curve $y = e^x$ at the point (1,e)

3 (b) (i) The graph of $y = e^x$ and $y = e^{-x} + 1$ are drawn below. Find the area between the curves from x = 2 to x = 3 and leave answer in terms of e

1 (ii) Show that the curves intersect when $e^{2x} - e^x - 1 = 0$



5 (c) If the shaded area below is rotated about the y-axis find the volume of the solid generated and leave answer in exact form.



Question 3

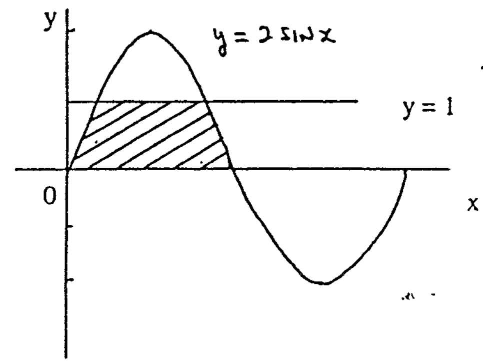
3 (a) Sketch $y = 3\cos 2x$ showing relevant points

2 (b) Find $\int \cot x\,dx$

(c) The line y = 1 meets the curve y = 2 sin x at A and B as shown in the diagram.

2 (i) Find the coordinates of A and B

5 (ii) Find the shaded area (in exact form)



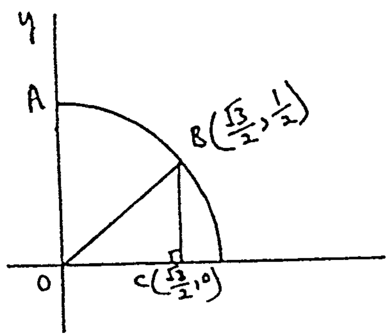
Question 4

1 (a) Find $\lim_{x \to 0} \frac{\sin 2x}{6x}$

(b) The diagram shows the first quadrant of the circle $x^2 + y^2 = 1$

2 (i) Find the value of $\angle AOB$

4 (ii) Find the exact area of AOCB



5 (c) A circle with circumference 120 cm has a chord cut off that subtends an angle of $40°$ at the centre. Draw a sketch. Find the length of the radius. Hence or otherwise find the length of the arc cut off by the chord.

Question 5

2 (a) Sketch $f(x) = x^2(x+1)(x-4)$ showing relevant points.

3 (b) $P(x) = x^3 + x - 2$
$A(x) = x - 1$

Divide P(x) by A(x) and leave answer in the form $P(x) = A(x)Q(x) + R(x)$

2 (c) Write down the most general cubic polymonial with the following properties ;
a monic polynomial with a double root at $x = 3$

5 (d) Given $f(x) = x^3 - 4x^2 + 5x - 1$

(i) show that f(x) has a zero between -1 and 1

(ii) taking $x = 0$ as a first approximation use Newtons method to find a better one.

(iii) show why $x = 1$ as a first approximation fails

Question 6

2 (a) When a polynomial P(x) is divided by $x^2 - 4$ the remainder is $2x - 1$. What is the remainder when P(x) is divided by $x + 2$.

5 (b) For $x^3 - 3x^2 - 4x + 2 = 0$. Find

(i) the sum of the roots

(ii) the product of the roots

(iii) the sum of the reciprocals of the roots

(iv) the sum of the squares of the roots

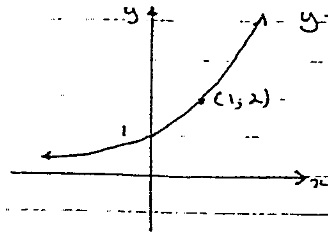
5 (c) The polynomial $f(x) = x^3 + ax^2 + bx + c$ has stationary points at $x = 1$ and $x = 5$ and also a double root. Find the two equations that would represent this.

4

Solutions S.G.H.S. Yr 12 3U Easter 1999

Q1.

a) i)



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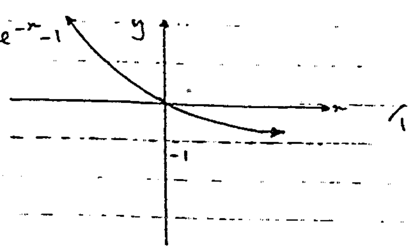
ii)



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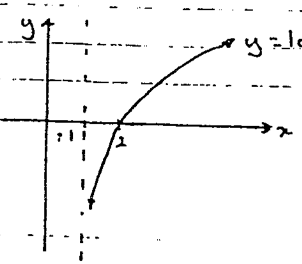
iii)



1

b) i) $y = e^{3x^2}$

$\frac{dy}{dx} = 6xe^{3x^2}$ 1

ii) $y = \frac{\log_e x}{x}$

$\frac{dy}{dx} = \frac{x(\frac{1}{x}) - \log_e x (1)}{x^2}$

$= \frac{1 - \log_e x}{x^2}$ 2

c) i) $\int e^{ax+b}\, dx = \frac{1}{a} e^{ax+b} + C$ 1

ii) $\int x e^{-x^2}\, dx$

If $y = e^{-x^2}$

then $\frac{dy}{dx} = -2xe^{-x^2}$

$\therefore \int -2xe^{-x^2} dx = e^{-x^2} + C_1$

$\therefore \int xe^{-x^2} dx = -\frac{1}{2} e^{-x^2} + C_2$ 1

iii) $\int \frac{6}{3x+5} dx = 2\int \frac{3}{3x+5} dx$

$= 2 \ln(3x+5) + C$ 1

d) $3 \log_4 2 = \log_4 x - \log_4 6$

$\log_4 2^3 + \log_4 6 = \log_4 x$

$\log_4 8 + \log_4 6 = \log_4 x$

$\log_4 (8 \times 6) = \log_4 x$

$\log_4 (48) = \log_4 x$

$\therefore x = 48$ 3

Q2

a) $y = e^x$

$\frac{dy}{dx} = e^x$

when $x = 1$

$m = e$

$y - y_1 = m(x - x_1)$

$y - e = e(x - 1)$

$y - e = ex - e$

$y = ex$ 3

b) i) $A = \int_2^3 (e^x - (e^{-x} + 1))\, dx$

$= \int_2^3 (e^x - e^{-x} - 1)\, dx$

$= [e^x + e^{-x} - x]_2^3$

$= (e^3 + e^{-3} - 3) - (e^2 + e^{-2} - 2)$

$= e^3 - e^2 + \frac{1}{e^3} - \frac{1}{e^2} - 1$ unit² 3

ii) $y = e^x$ ①, ② $y = e^{-x} + 1$

equating ① ② $e^x = e^{-x} + 1$

$\times e^x$) $e^{2x} = 1 + e^x$

c) $y = \log_e x$

ie $e^y = x$

$V = \pi \int_0^1 (e^y)^2\, dy$

$= \pi \int_0^1 e^{2y}\, dy$

$= \pi [\frac{1}{2} e^{2y}]_0^1$

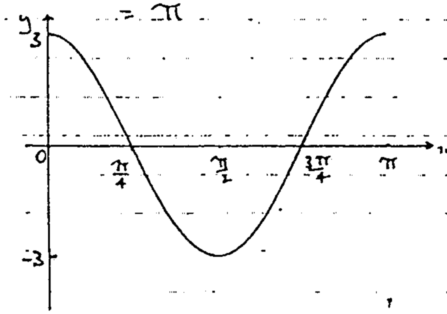
$= \frac{\pi}{2} [e^2 - 1]$

$= (\frac{\pi e^2}{2} - \frac{\pi}{2})$ units³ 4

Q3 a) $y = 3\cos 2x$

period $= \frac{2\pi}{2}$, amp. $= 3$

$= \pi$



b) $\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx$

$= \ln(\sin x) + C$

c) i) $2\sin x = 1$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$

ii) $V = 2\int_0^{\frac{\pi}{6}} 2\sin x\, dx + 1(\frac{5\pi}{6} - \frac{\pi}{6})$

$= 2[-2\cos x]_0^{\frac{\pi}{6}} + \frac{4\pi}{6}$

$= -4[\cos\frac{\pi}{6} - \cos 0] + \frac{2\pi}{3}$

$= -4[\frac{\sqrt{3}}{2} - 1] + \frac{2\pi}{3}$

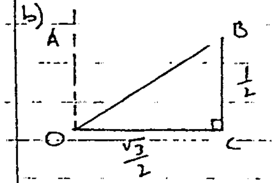
$= (-2\sqrt{3} + 4 + \frac{2\pi}{3})$ unit²

Q4 a) $\lim_{x \to 0} \frac{\sin 2x}{6x}$

$= \frac{1}{3} \lim_{x \to 0} \frac{\sin 2x}{2x}$

$= \frac{1}{3}(1)$

b)



i) $\tan \angle BOC = \frac{1}{2} \div \frac{\sqrt{3}}{2}$

$= \frac{1}{\sqrt{3}}$ ①

$\therefore \angle BOC = \frac{\pi}{6}$

$\angle AOB = \frac{\pi}{2} - \frac{\pi}{6}$

$= \frac{\pi}{3}$ ①

ii) Area = Area Sector AOB + Area Δ OBC

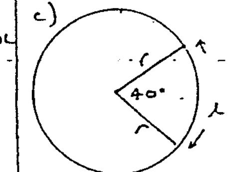
$= \frac{1}{2} r^2 \theta + \frac{BH}{2}$

$= \frac{1}{2}(1)^2 (\frac{\pi}{3}) + \frac{1}{2}(\frac{\sqrt{3}}{2})($

$= \frac{\pi}{6} + \frac{\sqrt{3}}{8}$

$= \frac{4\pi + 3\sqrt{3}}{24}$ unit²

c)



$C = 2\pi r$

$120 = 2\pi r$

$r = \frac{60}{\pi}$

$l = r\theta$ $\pi = 180°$, 40

$= \frac{60}{\pi} \times \frac{2\pi}{9}$

$= \frac{120}{9}$ cm

$= 13\frac{1}{3}$ cm

Q5

$f(x) = x^2(x+1)(x-4)$

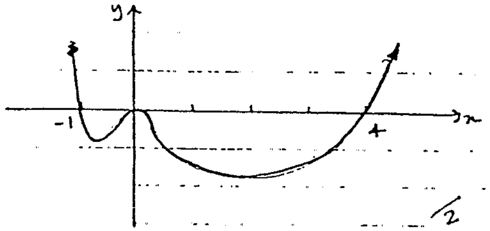
leading term x^4

" coefficient po

double root at $x = 0$

single roots at $x = -1$,

a)



/2

b)

$$
\begin{array}{r}
x^2 + x + 2 \\
x - 1 \overline{) x^3 \quad + x - 2} \\
\underline{x^3 - x^2} \\
x^2 + x \\
\underline{x^2 - x} \\
2x - 2 \\
\underline{2x - 2} \\
0
\end{array}
$$

/3

$\therefore x^3 + x - 2 = (x^2 + x + 2)(x - 1)$

c) $P(x) = (x-a)(x-3)^2$ /2

d) $f(x) = x^3 - 4x^2 + 5x - 1$

i) $f(-1) = (-1)^3 - 4(-1)^2 + 5(-1) - 1$

$= -1 - 4 - 5 - 1$

$= -11$

< 0

$f(1) = (1)^3 - 4(1)^2 + 5 - 1$

$= 1 - 4 + 5 - 1$

$= 1$

$> 0.$

$f(x)$ is continous and changes sign between $x = 1$ and $x = -1$ $\therefore$ a root lies between $x = 1$ and $x = -1$

ii) $f(x) = x^3 - 4x^2 + 5x - 1$

$f(0) = -1$

$f'(x) = 3x^2 - 8x + 5$

$f'(0) = 5$

$x_1 = a - \frac{f(a)}{f'(a)}$

$= 0 - \frac{-1}{5}$

$x_1 = \frac{1}{5}$

iii) $f'(x) = 3x^2 - 8x + 5$

$f'(1) = 3 - 8 + 5$

$= 0$

ie gradient of tangent is parallel to x axis. $\therefore$ tangent does not intersect with x axis $\therefore$ method fails

Q6 a) $P(x) = (x^2 - 4)Q(x) + 2x - 1$

$P(-2) = (0)Q(-2) + 2(-2) - 1$

$= -5$

/2

b) i) $\alpha + \beta + \gamma = \frac{-b}{a}$

$= 3$

ii) $\alpha\beta\gamma = \frac{-d}{a}$

$= -2$

/5

iii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta}$

$= \frac{c}{a} \div -2$

$= -4 \div -2$

$= 2$

iv) $\alpha^2 + \beta^2 + \gamma^2$

Now

$(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma) = \alpha^2 + \beta^2 + \gamma^2$

$3^2 - 2(-4) = \alpha^2 + \beta^2 + \gamma^2$

$17 = \alpha^2 + \beta^2 + \gamma^2$

c) $f(x) = x^3 + ax^2 + bx + c$

$f'(x) = 3x^2 + 2ax + b$

at a stny pt $f'(x) = 0$

ie at $x = 1$

$0 = 3 + 2a + b$

$-3 = 2a + b$ ①

at $x = !$

$0 = 75 + 2a + b$

$-75 = 10a + b$ ②

$-3 = 2a + b$ ①

②−① $-72 = 8a$

$a = -9$

$b = 15$

$\therefore f(x) = x^3 - 9x^2 + 15x + c$

If double root is at $x = 1$ (ie (1, 0)) then

$f(1) = 0$

$f(1) = 1 - 9 + 15 + c = 0$

$c = -7$

and $f(x) = x^3 - 9x^2 + 15x - 7$

If double root is at $x = 5$ (ie (5, 0)) then

$f(5) = 0$

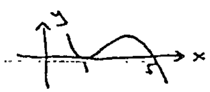
$f(5) = 125 - 225 + 75 + c$

$c = 25$

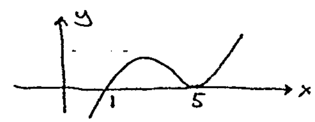
and $f(x) = x^3 - 9x^2 + 15x + 25$

/5

Note: $f(x) = (x-1)^2(x-5)$ has x intercepts 1, 5 ie there is no stat. pt at $x = 5$



or $f(x) = (x-1)(x-5)^2$



no stat pt at $x = 1$