Sydney Girls High School

2008
MATHEMATICS EXTENSION 1

YEAR 12
ASSESSMENT TASK 1
December 2007

Time Allowed: 75 minutes

Topics: Exponential and Logarithmic Functions, Integration and Locus

General Instructions:

- There are Four (4) Questions which are not of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- Start each Part on a new page.
- Write on one side of paper only.
- Diagrams are NOT to scale.
- Board-approved calculators may be used.
- Write your student number clearly at the top of each question and clearly number each question.

Question One (16 marks)

a) Solve $2^{3x-1} = 7$ giving answer to 2 decimal places. 2

b) Find $\frac{dy}{dx}$ if

i) $y = xe^{5x}$ 2

ii) $y = \frac{\ln(x+1)}{e^x}$ 2

c) A parabola is formed by graphing $y = \frac{x^2}{8}$

i) Find the coordinates of the focus and the equation of the directrix. 2

ii) Show the tangent at the point (4,2) on this parabola is $y = x - 2$ 2

iii) Find the area bounded by the parabola $y = \frac{x^2}{8}$, the tangent $y = x - 2$ and the y-axis. 3

d) If $y = log_{10}x$

i) Complete the table to two decimal points 1

X	1	2	3	4	5
y	0	0.30	0.48		

ii) Use Simpson's rule with five function values to find an approximation for $\int_1^5 log_{10}x\,dx$ to one decimal place. 2

Question Two (19 marks)

a) Find

i) $\int e^{3x}dx$ 1

ii) $\int \frac{x^2+x-1}{x}dx$ 2

iii) $\int \frac{3x}{x^2-1}dx$ 2

b) i) Show the equation of the locus of points that are twice the distance from the point A(-2,-1) as they are from the point B(4,2) is 3

$$3x^2 - 36x + 3y^2 - 18y + 75 = 0$$

ii) Give the coordinates of the centre of this circle and the length of its radius. 2

c) For the curve $y = \sqrt{x+1}$

i) Find the area between this curve and the x-axis between $x = 0$ *and* $x = 8$ 3

ii) Find the area between this curve and the y-axis between $y = 0$ *and* $y = 3$ 3

d) Simplify $2\log_6 3 + 2\log_6 2$ 3

Question Three (18 marks)

a) Find the volume of the solid formed by rotating the curve $y = \frac{1}{\sqrt{3x+1}}$ about the x-axis from $x = 0$ to $x = 2$. Give exact value. 3

b) Find the equation of a parabola with directrix $x = -4$ and vertex (0,2). 2

c) For $y = e^x - x$

i) Find any stationary points and determine their nature. 3

ii) On the same coordinate plane draw the graphs of $y = -x$, $y = e^x$ and $y = e^x - x$ showing all essential features. 3

d) i) Sketch the curve $y = x(x-1)(x+2)$. (without using calculus) 1

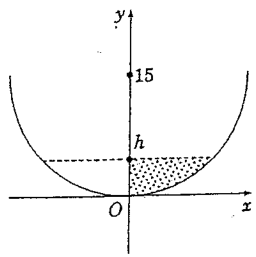
ii) Find the total area bounded by this curve and the x-axis. 3

e) Find the exact value of a so $\int_0^a e^{2x}dx = 1$ 3

Question Four (12Marks)

a) The diagram shows the lower half of the circle $x^2 + (y - 15)^2 = 15^2$ The shaded area in this diagram is bounded by the semicircle, the line $y = h$, and the y-axis.

Show that the volume V formed when the shaded area is rotated around the y-axis is given by $v = 15\pi h^2 - \frac{\pi h^3}{3}$ 3



b) Find the equations of the locus of points that are equidistant from the lines $y = 1$ and $5x - 12y + 1 = 0.$ 4

c) i) Show $\frac{2}{x-1} - \frac{1}{x+2} = \frac{x+5}{x^2+x-2}$ 1

ii) Use part i) to show $\int_2^3 \frac{x+5}{x^2+x-2} dx = ln3.2$ 4

The end

Solution's 2007/2008 Ext. 1 Year 12 Assessment 1

Question One

a) $2^{3x-1} = 7$, $(3x-1)\ln 2 = \ln 7$, $3x - 1 = \frac{\ln 7}{\ln 2}$, $3x = 1 + \frac{\ln 7}{\ln 2}$

$\therefore x = \frac{1}{3}\left(1 + \frac{\ln 7}{\ln 2}\right) \doteq 1.27$ (to 2 d.p.) (2)

b) i) $y = xe^{5x}$, $y' = x(5)e^{5x} + e^{5x}(1) = 5xe^{5x} + e^{5x}$ (2)

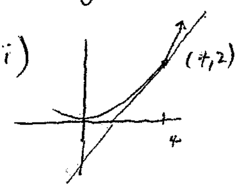
ii) $y = \frac{\ln(x+1)}{e^x}$, $y' = \frac{e^x\left(\frac{1}{x+1}\right) - e^x \ln(x+1)}{(e^x)^2} = \frac{\frac{e^x}{x+1} - e^x \ln(x+1)}{e^{2x}}$ (2)

c) i) $8y = x^2$ $\therefore 4a = 8$, $a = 2$ vertex $(0,0)$

$\therefore$ focus at $(0,2)$ directrix $x = -2$ (2)

ii) $y = x^2/8$ $y' = x/4$ when $x = 4$ $y' = m_{tangent} = 1$ (2)

$\therefore y - 2 = x - 4$ $\therefore y = x - 2$

iii) 

$A = \int_0^4 \frac{x^2}{8} - (x-2)\,dx = \int_0^4 \frac{x^2}{8} - x + 2\,dx$

$= \left[\frac{x^3}{24} - \frac{x^2}{2} + 2x\right]_0^4 = \left(\frac{64}{24} - \frac{16}{2} + 8\right) - (0)$

$= 8/3 - 8 + 8 = 8/3$ (3) 2

d) i)

x	4	5
y	.60	.70

(1)

ii) $A \doteq \frac{1}{3}(0 + 4(0.3) + 0.48) + \frac{1}{3}(0.48 + 4(.6) + 0.7)$

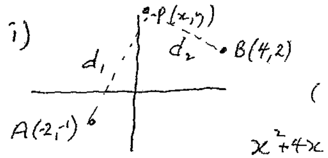
$\doteq 1.8$ (to one d.p.) (2) 1

Question Two

a) i) $\int e^{3x}\,dx = \frac{1}{3}\int 3e^{3x}\,dx = \frac{1}{3}e^{3x} + C$ ✓ (1)

ii) $= \int x + 1 - \frac{1}{x}\,dx = x^2/2 + x - \ln x + C$ ✓✓ (2)

iii) $= 3\int \frac{x}{x^2-1}\,dx = \frac{3}{2}\int \frac{2x}{x^2-1}\,dx = \frac{3}{2}\ln(x^2-1) + C$ ✓✓ (2)

b) i) 

$d_1^2 = 4d_2^2$ Show

$(x+2)^2 + (y+1)^2 = 4[(x-4)^2 + (y-2)^2]$

$x^2 + 4x + 4 + y^2 + 2y + 1 = 4(x^2 - 8x + 16 + y^2 - 4y + 4)$ ✓

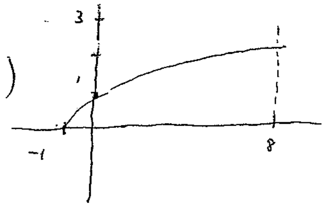
$x^2 + 4x + y^2 + 2y + 5 = 4x^2 - 32x + 4y^2 - 16y + 80$ ✓ (3)

$0 = 3x^2 - 36x + 3y^2 - 18y + 75$ ✓

ii) $3x^2 - 36x + 3y^2 - 18y + 75 = 0$ $(\div 3)$

$x^2 - 12x + y^2 - 6y + 25 = 0$, $x^2 - 12x + 36 + y^2 - 6y + 9 = -25 + 36 + 9$

$(x-6)^2 + (y-3)^2 = 20$ $\therefore$ centre $= (6,3)$ and radius $= \sqrt{20}$ ✓ ✓ (2)

c) i) 

$A = \int_0^8 (x+1)^{1/2}\,dx = \left.\frac{2(x+1)^{3/2}}{3}\right]_0^8$ ✓✓

$= \frac{2(9)^{3/2}}{3} - \frac{2}{3} = 17\frac{1}{3}$ (3)

ii) $y^2 = x + 1$ $A = \left|\int_0^1 y^2 - 1\,dx\right| + \int_1^3 y^2 - 1\,dx$ ✓

$x = y^2 - 1$

$A = \left|\left. y^3/3 - y\right]_0^1\right| + \left. y^3/3 - y\right]_1^3$ ✓ (3) 1

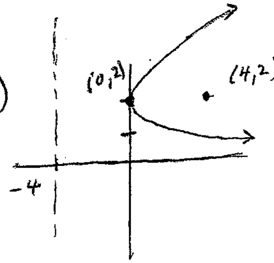
$= \left|\frac{1}{3} - 1\right| + (9 - 3) - \left(\frac{1}{3} - 1\right) = 2/3 + 6 + 2/3 = 7\frac{1}{3}$ ✓

d) $2\log_6 3 + 2\log_6 2 = 2(\log_6 3 + \log_6 2) = 2(\log_6 6) = 2(1) = 2$ ✓✓ (3)

Question 3.

a) $V = \pi \int_0^2 \left(\frac{1}{\sqrt{3x+1}}\right)^2 dx = \pi \int_0^2 \frac{1}{3x+1}\,dx = \frac{\pi}{3}\int_0^2 \frac{3}{3x+1}\,dx$

$= \frac{\pi}{3}\left[\ln(3x+1)\right]_0^2 = \frac{\pi}{3}\left[\ln 7 - 0\right] = \frac{\pi \ln 7}{3}$ (3)

b) 

focal length $= 4$ vertex $(0,2)$

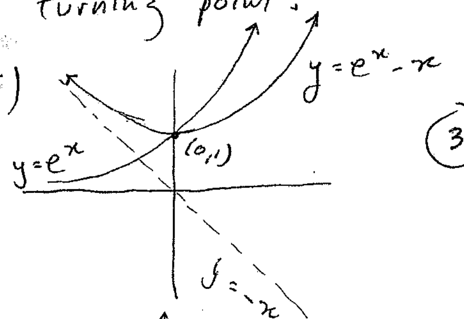
$\therefore$ using $4a(x-k) = (y-h)^2 \quad (h,k) = (0,2)$

$16(x-0) = (y-2)^2 \therefore 16x = (y-2)^2$ (2) 1

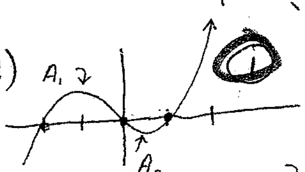
c) i) $y = e^x - x$, $y' = e^x - 1$, $e^x - 1 = 0$, $e^x = 1 \therefore x = 0$

$\therefore (0,1)$ is a Stationary Point. (3)

$y'' = e^x$ when $x = 0$ $y'' = 1 \therefore (0,1)$ is a minimum turning point.

ii) 

(3) 2

d) 

(1)

$A = A_1 + A_2 = \int_{-2}^0 x^3 + x^2 - 2x\,dx + \left|\int_0^1 x^3 + x^2 - 2x\,dx\right|$

$= \left.x^4/4 + x^3/3 - x^2\right]_{-2}^0 + \left|\left. x^4/4 + x^3/3 - x^2\right]_0^1\right| = -(4 - 8/3 - 4) + \left|\frac{1}{4} + \frac{1}{3} - 1\right|$

$= 8/3 + 5/12 = \frac{32}{12} + \frac{5}{12} = \frac{37}{12} = 3\frac{1}{12}$ (3) 2

Question 3 (cont)

e) $\int_0^a e^{2x}\,dx = \left.\frac{1}{2}e^{2x}\right]_0^a = \frac{1}{2}e^{2a} - \frac{1}{2}e^0 = \frac{1}{2}e^{2a} - \frac{1}{2}$

As $\frac{1}{2}e^{2a} - \frac{1}{2} = 1$, $\frac{1}{2}e^{2a} = \frac{3}{2}$, $e^{2a} = 3$

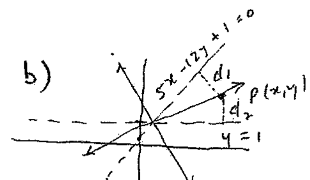
$\therefore 2a = \ln 3 \quad \therefore a = \ln 3/2$ (3)

Question 4.

a) $x = \sqrt{15^2 - (y-15)^2} = \sqrt{15^2 - (y^2 - 30y + 15^2)} = \sqrt{30y - y^2}$

$V = \pi \int_0^h \left[(30y - y^2)^{\frac{1}{2}}\right]^2 dy = \pi \int_0^h 30y - y^2\,dy = \pi\left[15y^2 - y^3/3\right]_0^h$

$= \pi\left[15h^2 - h^3/3\right] = 15\pi h^2 - \pi h^3/3$ (3)

b) 

$d_1 = \left|\frac{5x - 12y + 1}{\sqrt{5^2 + 12^2}}\right| = \frac{|5x - 12y + 1|}{13}$

$d_2 = y - 1$

$\therefore \frac{|5x - 12y + 1|}{13} = y - 1 \quad \therefore 5x - 12y + 1 = 13y - 13$

$5x - 25y + 14 = 0$ ✓

or $5x - 12y + 1 = -13y + 13$

$5x + y - 12 = 0$ ✓

(4) 2

c) i) $\frac{2}{x-1} - \frac{1}{x+2} = \frac{2x + 4 - x + 1}{x^2 + x - 2} = \frac{x + 5}{x^2 + x - 2}$ (1)

ii) $\int_2^3 \frac{2}{x-1} - \frac{1}{x+2}\,dx = \left.2\ln(x-1) - \ln(x+2)\right]_2^3 = \left.\ln\frac{(x-1)^2}{x+2}\right]_2^3$

$= \ln\frac{4}{5} - \ln\frac{1}{4} = \ln\frac{4}{5} \times \frac{4}{1} = \ln\frac{16}{5} = \ln 3.2$ (4)