SYDNEY GIRLS HIGH SCHOOL

2011

HSC ASSESSMENT TASK 2

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Writing time – 90 minutes
- Board-approved calculators may be used
- Diagrams are NOT to scale
- A table of standard integrals is provided at the back of this paper
- Topics – Complex Numbers

Total marks – 75

- Attempt Questions 1-3
- All questions are of equal value
- Start each question on a new page
- Write on one side of the paper only
- Write using black or blue pen
- All necessary working should be shown in every question

Name : ______________________________

Teacher : ______________________________

QUESTION 1 (25 Marks)

(a) If $\omega = 3-4i$ and $z=-1+3i$, express the following in the form $a+ib$ where a and b are real numbers.

(i) $\omega + 2z$ — 2

(ii) $z - \bar{z}$ — 2

(iii) $\dfrac{\omega}{z}$ — 2

(iv) $\text{Im}\left[\omega^2 + iz\right]$ — 3

(b) (i) Find $\sqrt{6i-8}$ and express each answer in the form $x+iy$. — 4

(ii) Hence solve the equation $2z^2-(3+i)z+2=0$, expressing z in the form $x+iy$. — 2

(c) If ω is a complex cube root of unity (i.e. a root of $z^3=1$) :

(i) Prove that ω^2 is also a complex cube root of unity. — 2

(ii) Prove that $1+\omega+\omega^2=0$. — 2

(iii) Evaluate $\left(3\omega^2+3\omega^4\right)^3$. — 2

(d) Find all the complex numbers $z=a+ib$, where a and b are real, such that $|z|^2+5\bar{z}+10i=0$. — 4

- End of Question 1 -

QUESTION 2 (25 Marks)

(a) $z_1 = 1 + i\sqrt{3}$ and $z_2 = 1 - i$ are two complex numbers.

(i) Express z_1 , z_2 and $\frac{z_1}{z_2}$ in modulus/argument form. 3

(ii) Find the smallest positive integer n such that $\frac{z_1^{\,n}}{z_2^{\,n}}$ is imaginary. 2

(iii) For this value of n, write the value of $\frac{z_1^{\,n}}{z_2^{\,n}}$ in the form bi where b is a real number. 1

(b) Find the Cartesian equation of the locus of z if :

(i) $|z-3i| = |z+6|$ 3

(ii) $2|z| = z + \bar{z} + 4$ 2

(iii) $\text{Re}\left(\frac{z-i}{z+1}\right) = 0$ 3

(c) (i) Write down the roots of the equation $z^5 = -1$ in modulus-argument form. 2

(ii) Plot the roots on an Argand diagram and sketch the polygon which has the roots as its vertices. 1

(iii) Find the perimeter of the polygon formed in (ii). Give your answer correct to 2 decimal places. 2

(iv) Using your answers from (i), express $z^5 + 1$ in terms of real linear and real quadratic factors. 2

- Question 2 continues on page 5 -

QUESTION 2 (continued)

(d) (i) Express $z = 1 + i$ in modulus/argument form. 1

(ii) Hence show that $z^9 = 16z$. 1

(iii) Hence express $(1+i)^9 + (1-i)^9$ in the form $a + ib$ where a and b are real. 2

- End of Question 2 -

QUESTION 3 (25 Marks)

(a) (i) If $z = \cos\theta + i\sin\theta$, show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$. 2

(ii) Hence show that $\cos^4\theta = \dfrac{1}{8}\left(\cos 4\theta + 4\cos 2\theta + 3\right)$. 3

(iii) Hence find $\int\left(8\cos^4\theta - 3\right)d\theta$. 2

(b) (i) On an Argand diagram shade the region where both $|z-1-i| \le \sqrt{2}$ and $0 \le \arg z \le \dfrac{\pi}{2}$. 3

(ii) Find the exact area of the shaded region. 2

(c) (i) Use De Moivre's Theorem to express $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$. 2

(ii) Hence show that $\tan 4\theta = \dfrac{4t - 4t^3}{1 - 6t^2 + t^4}$ where $t = \tan\theta$. 1

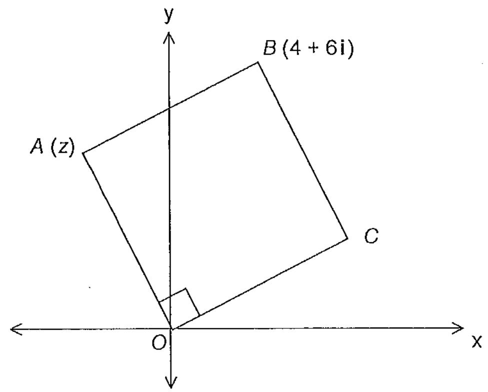
(iii) By first solving the equation $\tan 4\theta = 1$ for $0 \le \theta \le 2\pi$, solve the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$. 3

(iv) Hence find the value of $\tan\dfrac{\pi}{16} \times \tan\dfrac{3\pi}{16} \times \tan\dfrac{5\pi}{16} \times \tan\dfrac{7\pi}{16}$. 1

- Question 3 continues on page 7 -

QUESTION 3 (continued)

(d) $OABC$ is a square. A represents the complex number z. B represents the complex number $4+6i$.



(i) Write down the complex number represented by C in terms of z. 1

(ii) Hence, or otherwise, find z in the form of $x+iy$. 2

(e) Find the maximum value for $|z|$ where z satisfies the condition 3

$\left|z - \dfrac{3}{z}\right| = 5$.

- End of paper -

Question 1. Ext 2 Task 2. 2011

a. $w = 3-4i \qquad z = -1+3i$

i. $w+2z = 3-4i+2(-1+3i)$
$= 3-4i-2+6i$
$= 1+2i$

ii. $z-\bar{z} = 2i\times \text{Im}(z)$
$= 2i\times 3$

iii. $\frac{w}{z} = \frac{3-4i}{-1+3i}\times\frac{-1-3i}{-1-3i}$
$= \frac{-3-9i+4i+12i^2}{1-9i^2}$
$= \frac{-15-5i}{10}$
$= -\frac{3}{2}-\frac{1}{2}i$

iv. $\text{Im}(w^2+iz) = \text{Im}\left\{(3-4i)^2+i(-1+3i)\right\}$
$= \text{Im}(9-24i+16i^2-i+3i^2)$
$= \text{Im}(9-25i-19)$
$= \text{Im}(-10-25i)$
$= -25$

b.

i. $\sqrt{6i-8} = a+ib$
$6i-8 = a^2+2abi-b^2$
Equating real and imaginary parts:
$a^2-b^2 = -8 \rightarrow (1)$
$2ab = 6$
$b = \frac{3}{a} \rightarrow (2)$
Sub (2) into (1):
$a^2-\left(\frac{3}{a}\right)^2 = -8$
$a^4+8a^2-9 = 0$
$\left(a^2+9\right)\left(a^2-1\right) = 0$
a is real $\therefore\ a = \pm 1$
Sub $a = \pm 1$ into (2):
$b = \pm 3$
$\therefore \sqrt{6i-8} = \pm(1+3i)$

ii. $2z^2-(3+i)z+2 = 0$
$z = \frac{(3+i)\pm\sqrt{(3+i)^2-4\times 2\times 2}}{4}$
$= \frac{(3+i)\pm\sqrt{9+6i+i^2-16}}{4}$
$= \frac{(3+i)\pm\sqrt{6i-8}}{4}$
$= \frac{(3+i)\pm(1+3i)}{4}$
$z = \frac{4+4i}{4}$ or $z = \frac{2-2i}{4}$
$= 1+i \qquad = \frac{1}{2}-\frac{1}{2}i$

c.

i. $z^3 = 1$
$z = cis\left(\frac{2k\pi}{3}\right) \qquad k = 0,1,2$
$= 1,\ cis\frac{2\pi}{3},\ cis\frac{4\pi}{3}$
let $w = cis\frac{2\pi}{3}$
$w^2 = \left(cis\frac{2\pi}{3}\right)^2$
$= cis\frac{4\pi}{3}$ which is also a root.

ii. $1+w+w^2 = \frac{1(w^3-1)}{w-1} \qquad \left(w^3 = 1\right)$
$= \frac{1(1-1)}{w-1}$
$= 0$

iii. $(3w^2+3w^4)^3 = \left[3w^2(1+w^2)\right]^3$
$= \left[3w^2(-w)\right]^3$
$= (-3)^3$
$= -27$

d. $|z|^2+5\bar{z}+10i = 0$
$a^2+b^2+5a-5bi+10i = 0$
$a^2+b^2+5a+i(-5b+10) = 0$
Equating real and imaginary parts:
$-5b+10 = 0$
$b = 2$
$a^2+b^2+5a = 0$
$a^2+4+5a = 0$
$(a+4)(a+1) = 0$
$a = -4,\ -1$
$\therefore z = -1+2i,\ -4+2i$

2 (a) (i) $|z_1| = \sqrt{1^2+(\sqrt{3})^2}$

$= 2$

$\tan\theta = \frac{\sqrt{3}}{1}$

$\theta = \frac{\pi}{3}$

$\therefore z_1 = 2\,\text{cis}\,\frac{\pi}{3}$ ✓

$|z_2| = \sqrt{1^2+(-1)^2}$

$= \sqrt{2}$

$\tan\theta = \frac{-1}{1}$

$\theta = -\frac{\pi}{4}$

$\therefore z_2 = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$ ✓

$\frac{z_1}{z_2} = \frac{2}{\sqrt{2}}\,\text{cis}\left(\frac{\pi}{3} - -\frac{\pi}{4}\right)$

$= \sqrt{2}\,\text{cis}\,\frac{7\pi}{12}$ ✓

(ii) $\left(\frac{z_1}{z_2}\right)^n = r\,\text{cis}\left(\frac{k\pi}{2}\right)$ k an integer

$(\sqrt{2})^n\,\text{cis}\,\frac{7n\pi}{12} = r\,\text{cis}\left(\frac{k\pi}{2}\right)$ ✓

$\frac{7n\pi}{12} = \frac{k\pi}{2}$

$7n = 6k$

$\therefore n = 6$ ✓

(iii) $b = (\sqrt{2})^6 \sin\left(\frac{7\times 6\times\pi}{12}\right)$

$= 8\sin\frac{3\pi}{2}$

$= -8$ ✓

(b) (i) $\sqrt{x^2+(y-3)^2} = \sqrt{(x+6)^2+y^2}$ ✓

$x^2+y^2-6y+9 = x^2+12x+36+y^2$

$0 = 12x+6y+27$

$4x+2y+9 = 0$ ✓

(ii) $2\sqrt{x^2+y^2} = x+iy+x-iy+4$

$4(x^2+y^2) = (2x+4)^2$ ✓

$4x^2+4y^2 = 4x^2+16x+16$

$4y^2 = 16x+16$ ✓

$y^2 = 4x+4$

(iii) $\frac{z-i}{z+1} = \frac{x+iy-i}{x+iy+1}$ ✓

$= \frac{x+i(y-1)}{x+1+iy} \times \frac{x+1-iy}{x+1-iy}$

$= \frac{x(x+1) - ixy + i(y-1)(x+1) + y(y-1)}{(x+1)^2+y^2}$ ✓

$\text{Re}\left(\frac{z-i}{z+1}\right) = 0$

$\frac{x(x+1)+y(y-1)}{(x+1)^2+y^2} = 0$

$x(x+1)+y(y-1) = 0$ ✓

$x^2+x+\left(\frac{1}{2}\right)^2+y^2-y+\left(-\frac{1}{2}\right)^2 = \frac{1}{2}$

$\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{1}{2}$

(c) (i) $r^5\,\text{cis}\,5\theta = -1$ ✓

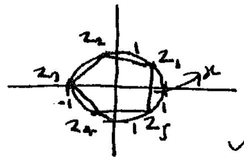
$r^5 = 1 \quad 5\theta = \pi, 3\pi, 5\pi, 7\pi, 9\pi$

$r = 1 \quad \theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$

$z_1 = \text{cis}\,\frac{\pi}{5},\ z_2 = \text{cis}\,\frac{3\pi}{5},\ z_3 = \text{cis}\,\pi$

$z_4 = \text{cis}\,\frac{-3\pi}{5},\ z_5 = \text{cis}\,\frac{-\pi}{5}$ ✓

(ii)



✓

(iii) $x^2 = 1^2+1^2-2\times1\times1\times\cos\frac{2\pi}{5}$

$x = \sqrt{2-2\cos\frac{2\pi}{5}}$ ✓

$P = 5\left(2-2\cos\frac{2\pi}{5}\right)^{\frac{1}{2}}$

$= 5.87|778\ldots$

$= 5.88$ m ✓

(iv) $z^5+1 = (z+1)\left(z-\text{cis}\,\frac{\pi}{5}\right)\left(z-\text{cis}\,\frac{-\pi}{5}\right)\left(z-\text{cis}\,\frac{3\pi}{5}\right)\left(z-\text{cis}\,\frac{-3\pi}{5}\right)$ ✓

$= (z+1)\left(z-\cos\frac{\pi}{5}-i\sin\frac{\pi}{5}\right)\left(z-\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)\left(z-\cos\frac{3\pi}{5}-i\sin\frac{3\pi}{5}\right)\left(z-\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}\right)$

$= (z+1)\left(z^2-2z\cos\frac{\pi}{5}+\cos^2\frac{\pi}{5}+\sin^2\frac{\pi}{5}\right)\left(z^2-2z\cos\frac{3\pi}{5}+\cos^2\frac{3\pi}{5}+\sin^2\frac{3\pi}{5}\right)$ ✓

$= (z+1)\left(z^2-2z\cos\frac{\pi}{5}+1\right)\left(z^2-2z\cos\frac{3\pi}{5}+1\right)$ ✓

(d) (i) $|z| = \sqrt{1^2+1^2}$

$= \sqrt{2}$

$\tan\theta = \frac{1}{1}$

$\theta = 45°$

$z = \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$ ✓

(ii) $z^9 = (\sqrt{2})^9\,\text{cis}\,\frac{9\pi}{4}$

$= 16\sqrt{2}\,\text{cis}\,\frac{\pi}{4}$ ✓

$= 16z$

Q.E.D.

(iii) $z^9 + \bar{z}^9$

$= 16z + \overline{z^9}$ ✓

$= 16z + \overline{16z}$

$= 16(1+i) + 16(1-i)$

$= 32$ ✓

QUESTION 3:

a) (i)

$$z^n+\frac{1}{z^n}=z^n+z^{-n}$$
$$=(\cos\theta+i\sin\theta)^n+(\cos\theta+i\sin\theta)^{-n}$$
$$=\cos n\theta+i\sin n\theta+\cos(-n\theta)+i\sin(-n\theta)$$
$$=\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta$$
$$=2\cos n\theta$$

(ii)

$$\left(z+\frac{1}{z}\right)^4=z^4+4z^3\cdot\frac{1}{z}+6z^2\cdot\frac{1}{z^2}+4z\cdot\frac{1}{z^3}+\frac{1}{z^4}$$
$$(2\cos\theta)^4=\left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6$$
$$16\cos^4\theta=2\cos 4\theta+8\cos 2\theta+6$$
$$\cos^4\theta=\frac{1}{8}\cos 4\theta+\frac{1}{2}\cos 2\theta+\frac{3}{8}$$
$$=\frac{1}{8}(\cos 4\theta+4\cos 2\theta+3)$$

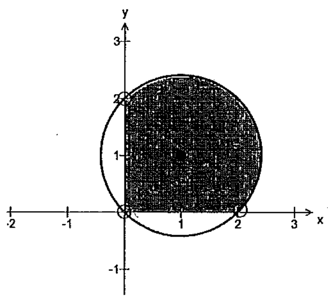
(iii)

$$\cos^4\theta=\frac{1}{8}(\cos 4\theta+4\cos 2\theta+3)$$
$$8\cos^4\theta=\cos 4\theta+4\cos 2\theta+3$$
$$8\cos^4\theta-3=\cos 4\theta+4\cos 2\theta$$
$$\int(8\cos^4\theta-3)d\theta=\int(\cos 4\theta+4\cos 2\theta)d\theta$$
$$=\frac{1}{4}\sin 4\theta+2\sin 2\theta+C$$

b) (i)

$$|z-1-i|\le\sqrt{2}$$
$$|z-\{1+i\}|\le\sqrt{2}$$

Circle centre $(1,1)$ and radius $\sqrt{2}$ units



(ii) Required area = Area of semi-circle +Area of triangle

$$\text{Area}=\frac{1}{2}\times\pi\times\left(\sqrt{2}\right)^2+\frac{1}{2}\times 2\times 2$$
$$=(\pi+2)\text{u}^2$$

c) (i)

$$(\cos\theta+i\sin\theta)^4=\cos^4\theta+4\cos^3\theta(i\sin\theta)+6\cos^2\theta(i\sin\theta)^2+4\cos\theta(i\sin\theta)^3+(i\sin\theta)^4$$
$$\cos 4\theta+i\sin 4\theta=\cos^4\theta+i4\cos^3\theta\sin\theta-6\cos^2\theta\sin^2\theta-i4\cos\theta\sin^3\theta+\sin^4\theta$$
$$\cos 4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta$$
$$\sin 4\theta=4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta$$

(ii)

$$\tan 4\theta=\frac{\sin 4\theta}{\cos 4\theta}$$
$$=\frac{4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta}{\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta}$$
$$=\frac{\frac{4\cos^3\theta\sin\theta}{\cos^4\theta}-\frac{4\cos\theta\sin^3\theta}{\cos^4\theta}}{\frac{\cos^4\theta}{\cos^4\theta}-\frac{6\cos^2\theta\sin^2\theta}{\cos^4\theta}+\frac{\sin^4\theta}{\cos^4\theta}}$$
$$=\frac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$$
$$=\frac{4t-4t^3}{1-6t^2+t^4}\quad(\text{using } t=\tan\theta)$$

(iii)

$$\tan 4\theta=1$$
$$4\theta=\frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4},\frac{13\pi}{4},\frac{17\pi}{4},\frac{21\pi}{4},\frac{25\pi}{4},\frac{29\pi}{4}$$
$$\theta=\frac{\pi}{16},\frac{5\pi}{16},\frac{9\pi}{16},\frac{13\pi}{16},\frac{17\pi}{16},\frac{21\pi}{16},\frac{25\pi}{16},\frac{29\pi}{16}$$
$$x^4+4x^3-6x^2-4x+1=0$$

Let $x=\tan\theta$

$$\tan^4\theta+4\tan^3\theta-6\tan^2\theta-4\tan\theta+1=0$$

Let $t=\tan\theta$

$$t^4+4t^3-6t^2-4t+1=0$$
$$t^4-6t^2+1=4t-4t^3$$
$$\frac{4t-4t^3}{t^4-6t^2+1}=1$$
$$\tan 4\theta=1$$
$$\theta=\frac{\pi}{16},\frac{5\pi}{16},\frac{9\pi}{16},\frac{13\pi}{16}$$
$$x=\tan\frac{\pi}{16},\tan\frac{5\pi}{16},\tan\frac{9\pi}{16},\tan\frac{13\pi}{16}$$

(iv)

$$\tan\frac{\pi}{16}\times\tan\frac{5\pi}{16}\times\tan\frac{9\pi}{16}\times\tan\frac{13\pi}{16}=1 \quad \text{(product of roots)}$$

$$\tan\frac{\pi}{16}\times\tan\frac{5\pi}{16}\times\tan\left(\pi-\frac{7\pi}{16}\right)\times\tan\left(\pi-\frac{3\pi}{16}\right)=\tan\frac{\pi}{16}\times\tan\frac{5\pi}{16}\times-\tan\frac{7\pi}{16}\times-\tan\frac{3\pi}{16}$$

$$=\tan\frac{\pi}{16}\times\tan\frac{5\pi}{16}\times\tan\frac{7\pi}{16}\times\tan\frac{3\pi}{16}$$

So $\tan\frac{\pi}{16}\times\tan\frac{3\pi}{16}\times\tan\frac{5\pi}{16}\times\tan\frac{7\pi}{16}=1$

d) (i) $C=-iz$

(ii) $\overrightarrow{OA}+\overrightarrow{OC}=\overrightarrow{OB}$

$$z-iz=4+6i$$

$$z(1-i)=4+6i$$

$$z=\frac{4+6i}{1-i}\times\frac{1+i}{1+i}$$

$$=\frac{4+10i-6}{2}$$

$$=\frac{-2+10i}{2}$$

$$=-1+5i$$

(iii)

$$\left|z-\frac{3}{z}\right|=5$$

Using the triangular inequality $|z_1-z_2|\geq|z_1|+|z_2|$

$$\left|z-\frac{3}{z}\right|\geq|z|-\left|\frac{3}{z}\right|$$

$$5\geq|z|-\left|\frac{3}{z}\right|$$

$$|z|-\left|\frac{3}{z}\right|\leq 5$$

$$|z|-\frac{3}{|z|}\leq 5$$

$$|z|^2-3\leq 5|z|$$

$$|z|^2-5|z|\leq 3$$

Completing the square:

$$|z|^2-5|z|+\left(-\frac{5}{2}\right)^2\leq 3+\left(-\frac{5}{2}\right)^2$$

$$\left(|z|-\frac{5}{2}\right)^2\leq\frac{37}{4}$$

$$|z|-\frac{5}{2}\leq\frac{\sqrt{37}}{2} \quad \text{since modulus} > 0$$

$$|z|\leq\frac{\sqrt{37}+5}{2}$$

Maximum value of $|z|=\frac{\sqrt{37}+5}{2}$