SYDNEY GIRLS HIGH SCHOOL

2007 Assessment Task 2

Monday, 5th March, 2007

MATHEMATICS

Year 12

Time allowed: 90 minutes

Total marks: 80

Topics: Probability, Sequences & Series, Quadratic Polynomials.

DIRECTIONS TO CANDIDATES:.

- Attempt all questions
- Questions are of equal value
- There are 5 questions with part marks shown in bold
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work
- Board approved calculators may be used
- Write on one side of the paper only

Question 1 (16 marks)

a) The first three terms of an arithmetic sequence are 5, 9 , 13. **3**
   1. Write down a formula for the nth term.
   2. Find the eleventh term
   3. How many terms are in the series if the last term of the series is 97?

b) The first term of an arithmetic series is 4 and the fifth term is four times the third term. **2**
Find the common difference.

c) The first two terms of an arithmetic sequence are -17, -14. **3**
   1. Write down the sum of the first n terms .
   2. Find the sum of the first twenty terms.
   3. What is the least value of n for which the sum of the first n terms is positive.

d) The sum of the first n terms of an arithmetic series is given by $S_n = n(2n+1)$. **2**
Find an expression for the n^{th} term.

e) Find the sum of the first 2000 terms of the series **2**
$1 - 2 + 3 - 4 + 5 - 6 + \ldots\ldots\ldots\ldots\ldots + (-1)^{n+1} \times n$.   -2000

f) Find the number of terms in the geometric sequence **2**
$\frac{4}{243}, \frac{4}{81}, \ldots\ldots, 36, 108$.

g) The third term of a geometric series is 54, and the sixth term is 2. Find **2**
   1. the common ratio;
   2. the first term.

Question 2 (16 marks)

a) The sequence is $\frac{1}{2}$, 1, 2, 4, ... is geometric. **2**
Find the sum of the first ten terms.
Give the answer as a rational number in its lowest terms.

b) An infinite geometric series has a first term of 8 and a limiting sum of 12. Calculate the common ratio. **2**

c) Express $0.\dot{4}\dot{7}$ as the sum of an infinite geometric progression. Hence express $0.\dot{4}\dot{7}$ as a simple fraction. **2**

d) Find the number which when added to each of 2, 6, 13 gives a set of three numbers in geometric progression. **2**

e) Rosie joins a superannuation fund by investing \$3000 at 9% p.a. compound interest at the beginning of each year for 28 years. **3**
Find the accumulated value of the investment after twenty-eight years. Write your answer correct to the nearest dollar.

f) When Melissa left school she borrowed \$15 000 to buy her first car. The interest rate on the loan was 18% p.a. reducible. The money is to be paid back in equal monthly instalments over 5 years. At the end of each month interest is added to the principle before the monthly instalment is deducted. **5**
Let the amount of each monthly payment be M dollars and the amount owing after n payments be A_n.

i. Write down the amount A_1 owing after one payment in terms of M.

ii. Show that the amount owing after two payments is
$$A_2 = 15000(1.015)^2 - M(1 + 1.015)$$

iii. Write down an expression for A_{60}

iv. Hence calculate the amount of each monthly instalment to the nearest dollar.

Question 3 (16 marks)

a) Natasha has four pairs of socks, each pair a different style. If she selects two socks at random, what is the probability that they form a matching pair? **1**

b) Comment briefly on the following statement, giving reasons for your view : **1**
"There are twelve teams in a football competition. The probability that a particular team will win is $\frac{1}{12}$".

c) A pair of dice are thrown together at random and the numbers 1 to 6 on each die are equally likely to appear. Find the probability that **5**

i) they both show a 6.
ii) they show a 1 and a 6.
iii) at least one of them shows a 1.
iv) they show a total of six.
v) the sum of the two numbers is at least 10.

d) One hundred tickets are to be sold in a raffle. Two different tickets are to be drawn out for first and second prizes respectively. Katie buys ten tickets. **4**

Find the probability that

i. Katie wins first prize
ii. Katie wins both prizes
iii. Katie wins neither prize
iv. Katie wins at least one prize.

e) Four metal disks numbered 1, 2, 3, 4 are placed in a bag. **3**
Two disks are selected at random and placed together on a table top to form a two digit number.

i) Draw a tree diagram to show the possible outcomes.
ii) Find the probability that the number formed is 21.
iii) Determine the probability that the number formed is divisible by 3.

f) On a destroyer there are two lines of defence against anti-aircraft attack. These are a surface-to-air missile and a 15 mm rapid firing gun. The probability of success in hitting an attacking aircraft with each line of defence is respectively 0.9 and 0.8. **2**
Find the probability of hitting an attacking aircraft before it penetrates both defences.

Question 4 (16 marks)

a) An urn contains 4 black and 3 white balls. Two balls are drawn at random and placed in a hat **3**

i) Draw a probability tree to show the possible outcomes. Write the probability on each branch.
ii) Find the probability that the hat contains two white balls.
iii) Find the probability that the hat contains a white and a black ball.

b) In a Year 12 class the probability that a student plays soccer is $P(S) = {}^3/_4$ and that a student plays cricket is $P(C) = {}^1/_3$. **2**
The probability that a student plays both Soccer and Cricket is $P(S \cap C) = {}^1/_8$.
Find the probability $P(S \cup C)$ that a student selected at random from the class plays either soccer or cricket or both.

c) In a group of 40 girls there are 29 girls who travel to school by train and 23 who travel by bus, while 7 travel by neither **3**

i) Draw a Venn diagram using the information above.
ii) What is the probability that a girl chosen at random travels by train and bus
iii) What is the probability that a girl chosen at random travels only by bus.

d) Draw a neat sketch of $y = x^2 + 2x - 8$ showing **4**

i) x intercepts
ii) y intercept
iii) axis of symmetry
iv) vertex

e) Use your graph in part d) to solve $x^2 + 2x - 8 \geq 0$ **2**

f) Find the discriminant of $2x^2 + 3x - 5$ and state whether the roots of the quadratic equation $2x^2 + 3x - 5 = 0$ are real or unreal. **2**

Question 5 (16 marks)

a) Without sketching determine whether the curve $y = 3x^2 - 4x + 5$ crosses the x-axis or not. **2**

b) Find all values of k for which the quadratic equation $kx^2 - 8x + k = 0$ has equal roots. **2**

c) For what values of m is the line $y = m(x-1)$ a tangent to the parabola $y = 2x^2$. **2**

d) The quadratic equation $2x^2 - x - 3 = 0$ has roots α and β **5**

i. calculate: $\alpha + \beta$
ii. calculate: $\alpha\beta$
iii. calculate: $\alpha^2 + \beta^2$
iv. calculate: $\alpha^2\beta^2$
v. find a quadratic equation which has roots $x = \alpha^2$ and $x = \beta^2$.

e) The roots of the quadratic equation $mx^2 + x + n = 0$ are 2 and -1. Find m and n. **2**

f) Find the value of j such that the roots of $x^2 + 7x + j = 0$ are reciprocals of each other. **1**

g) Find the values of k if the expression $kx^2 - 12x + 3k$ is positive definite. Give exact values. **2**

Year 12, Mathematics, March 07.

1 a) 5, 9, 13

$a = 5, d = 4$

i) $T_n = a + (n-1)d$ ①

$= 5 + (n-1)4$

$= 5 + 4n - 4$

$T_n = 4n + 1$

ii) $T_{11} = 4 \times 11 + 1$ ①

$= 45$

iii) $T_n = 97$

$4n + 1 = 97$ ①

$4n = 96$

$n = 24$

b) $T_1 = 4$, $a = 4$

$T_5 = 4 \cdot T_3$

$a + 4d = 4(a + 2d)$

$a + 4d = 4a + 8d$

$0 = 3a + 4d$ ②

$3 \times 4 + 4d = 0$

$4d = -12$

$d = -3.$

c) i) $-17, -14$

$a = -17, d = 3$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$= \frac{n}{2}[2(-17) + (n-1)3]$

$= \frac{n}{2}[-34 + 3n - 3]$ ①

$S_n = \frac{n}{2}[3n - 37]$

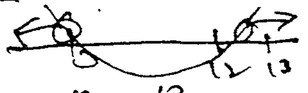
ii) $S_{20} = \frac{20}{2}[3 \times 20 - 37]$

$= 10 \times 23$ ①

$= 230$

iii) $S_n > 0$

$\frac{n}{2}(3n - 37) > 0$ ①



$n = 13.$

d) $S_n = n(2n+1) = 2n^2 + n$

$S_{n-1} = (n-1)(2(n-1)+1)$

$= (n-1)(2n-2+1)$

② $= (n-1)(2n-1)$

$= 2n^2 - n - 2n + 1$

$\therefore S_{n-1} = 2n^2 - 3n + 1$

$T_n = S_n - S_{n-1}$

$= 2n^2 + n - (2n^2 - 3n + 1)$

$= 2n^2 + n - 2n^2 + 3n - 1$

$\therefore T_n = 4n - 1$

e) $1 + 3 + 5 + \cdots + 1999$

$= \frac{n}{2}(a + l)$ $a = 1$, $l = 1999$, $n = 1000$

$= \frac{1000}{2}(1 + 1999)$

② $= 1\,000\,000$

$2 + 4 + 6 + \cdots + 2000$

$= \frac{n}{2}(a + l)$, $a = 2, d = 2, l = 2000$

$= \frac{1000}{2}(2 + 2000)$ $n = 1000$

$= 1\,001\,000$

Sum $= 1\,000\,000 - 1\,001\,000$

$= -1\,000.$

f) $\frac{4}{243}, \frac{4}{81}, \ldots, 36, 108$

$a = \frac{4}{243}, r = 3, T_n = 108$

$T_n = a \cdot r^{n-1}$

$= \frac{4}{243} \times 3^{n-1}$

$= \frac{4}{3^5} \times 3^{n-1}$

$T_n = 4 \times 3^{n-6}$

$4 \times 3^{n-6} = 108$ ②

$3^{n-6} = 27$

$3^{n-6} = 3^3$

$n = 9.$

1(g) i) $T_3 = 54$, $T_6 = 2$

$ar^2 = 54$

$ar^5 = 2$

$\frac{ar^5}{ar^2} = \frac{2}{54}$

$r^3 = \frac{1}{27}$ (2)

$r = \frac{1}{3}$

ii) $a \times (\frac{1}{3})^2 = 54$

$a = 54 \times 9$

$= 486$

Quest 2

a) $a = \frac{1}{2}$, $r = 2$

$S_n = \frac{a(r^n - 1)}{r - 1}$

$S_{10} = \frac{\frac{1}{2}(2^{10} - 1)}{2 - 1}$ (2)

$= \frac{1}{2} \times (1024 - 1)$

$= \frac{1}{2} \times 1023$

$\therefore S_{10} = 511\frac{1}{2}$.

b) $a = 8$, $\lim S = 12$.

$\frac{a}{1-r} = 12$

$\frac{8}{1-r} = 12$ (2)

$8 = 12 - 12r$

$12r = 4$

$\therefore r = \frac{1}{3}$

c) $0.\dot{4}\dot{7} = 0.47 + 0.0047 + 0.000047 + \dots$

$= \frac{a}{1-r}$, $a = 0.47$, $r = 0.01$

$= \frac{0.47}{1 - 0.01}$

$= \frac{0.47}{0.99} = \frac{47}{99}$ (2)

d) 2, 6, 13

g.p.: $2 + x$, $6 + x$, $13 + x$

$r = \frac{6+x}{2+x} = \frac{13+x}{6+x}$ (2)

$(6 + x)^2 = (2 + x)(13 + x)$

$36 + 12x + x^2 = 26 + 2x + 13x + x^2$

$36 - 26 = 15x - 12x$

$3x = 10$

$x = \frac{10}{3} = 3\frac{1}{3}$.

e) $P = 3000$, $r = 9\%$ pa $= 0.09$, $n = 28$

$Sum = 3000 \times 1.09^{28} + 3000 \times 1.09^{27} + \dots + 3000 \times 1.09$

$= 3000 (1.09 + 1.09^2 + 1.09^3 + \dots + 1.09^{28})$

$= 3000 \times \frac{a(r^n - 1)}{r - 1}$ $a = 1.09$, $r = 1.09$, $n = 28$

$= 3000 \times 1.09 \times \frac{(1.09^{28} - 1)}{1.09 - 1}$

$= 3000 \times 1.09 \times \frac{(1.09^{28} - 1)}{0.09}$ (3)

$= \$369\ 406$

f) $P = 15\,000$, $r = 18\%$ pa $= 1.5\%$ per month $= 0.015$.

$n = 5$ yrs $= 60$ months

i) $A_1 = 15000 \times 1.015 - M$. (1)

ii) $A_2 = (15000 \times 1.015 - M) \times 1.015 - M$

$= 15000 \times 1.015^2 - M \times 1.015 - M$ (1)

$= 15000 \times 1.015^2 - M(1 + 1.015)$

2 b)

2 iii) $A_{60} = 15000 \times 1.015^{60} - M(1 + 1.015 + \dots + 1.015^{59})$

$= 15000 \times 1.015^{60} - M \times \frac{a(r^n - 1)}{r - 1}$

$= 15000 \times 1.015^{60} - M \times 1 \times \frac{(1.015^{60} - 1)}{1.015 - 1}$ (1)

$= 15000 \times 1.015^{60} - M \frac{(1.015^{60} - 1)}{0.015}$

iv) $A_{60} = 0$

$15000 \times 1.015^{60} - M \frac{(1.015^{60} - 1)}{0.015} = 0$ (2)

$M = \frac{15000 \times 1.015^{60} \times 0.015}{(1.015^{60} - 1)}$

$= \$225$.

Question 3:

i) After the 1st sock is chosen, only one sock will complete the matching pair.

$P(\text{matching pair}) = \frac{1}{7}$ (1)

) Teams do not have equal ability, hence they do not have an equal chance of winning. (1)

c i) $P(6 \,\&\, 6) = \frac{1}{36}$ (1)

ii) $P(1 \,\&\, 6) = P(1,6) + P(6,1)$

$= \frac{1}{36} + \frac{1}{36}$

$= \frac{1}{18}$ (1)

iii) $P(\text{at least one } 6) = \frac{11}{36}$ (1)

iv) $P(\text{total of } 6) = \frac{5}{36}$ (1)

v) $P(\text{sum} \geq 10) = \frac{1}{6}$ (1)

d i) $P(1^{st}\text{ prize}) = \frac{1}{10}$ (1)

ii) $P(\text{both}) = \frac{10}{100} \times \frac{9}{99}$

$= \frac{1}{110}$ (1)

iii) $P(\text{neither}) = \frac{90}{100} \times \frac{89}{99}$

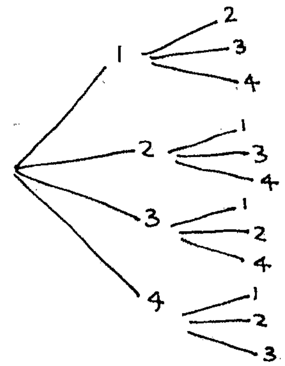
$= \frac{89}{110}$ (1)

iv) $P(\text{at least } 1) = 1 - \frac{89}{110}$

$= \frac{21}{110}$ (1)

e)

i)



1,2
1,3
1,4

2,1
2,3
2,4

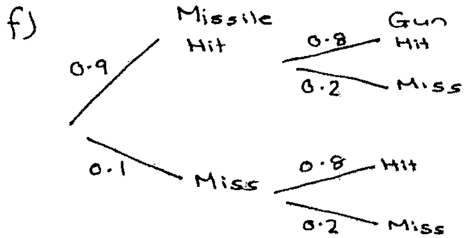
3,1
3,2
3,4

4,1
4,2
4,3 (1)

ii) $P(21) = \frac{1}{12}$ (1)

iii) $P(\text{div. by } 3) = \frac{4}{12} = \frac{1}{3}$ (1)

f)



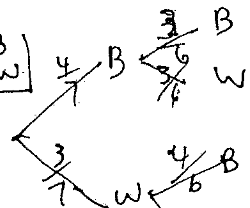
$P(\text{hit aircraft}) = 1 - P(\text{miss both})$

$= 1 - (0.1 \times 0.2)$

$= 0.98$ (2)

Quest 4

a) i) $\boxed{\begin{matrix} 4B \\ 3W \end{matrix}}$



(1)

ii) $P(WW) = \frac{3}{7} \times \frac{2}{6} = \frac{1}{7}$ (1)

iii) $P(WB \text{ or } BW) = \frac{4}{7} \times \frac{3}{6} + \frac{3}{7} \times \frac{4}{6} = \frac{24}{42}$

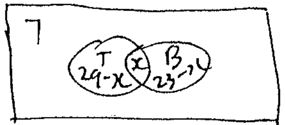
$= \frac{4}{7}$ (1)

M

b) $P(S)=\frac{3}{4}$, $P(C)=\frac{1}{3}$.
$P(S\cap C) = \frac{1}{8}$.

$P(S\cup C)=P(S)+P(C)-P(S\cap C)$
$=\frac{3}{4}+\frac{1}{3}-\frac{1}{8}$
$=\frac{18+8-3}{24}$
$=\frac{23}{24}$. (2)

c) i)



(1)

ii) $7+(29-x)+(23-x)+x=40$
$59-x=40$
$\therefore x=19$
$P(\text{train + bus})=\frac{19}{40}$ (1)

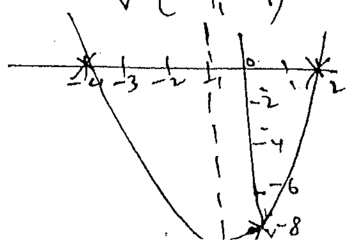
iii) $23-x=23-19=4$
$P(\text{bus only})=\frac{4}{40}=\frac{1}{10}$. (1)

d) i) $y=x^2+2x-8$
$y=(x+4)(x-2)$
X int $= -4$ or 2

ii) $x=0,\ y=-8$
Y-int $=-8$

iii) Axis: $x=\frac{-4+2}{2}$
$=\frac{-2}{2}$
$\therefore x=-1$

iv) $x=-1,\ y=(-1)^2+2(-1)-8$
$=1-2-8$
$=-9$
$V(-1,-9)$



(4)

e) $x^2+2x-8\geq 0$
$x\leq -4$ or $x\geq 2$
(2)

f) $2x^2+3x-5$
$a=2,\ b=3,\ c=-5$
$\Delta=b^2-4ac$
$=9-4\times 2\times -5$
$=9+40$
$\therefore \Delta = 49$ (1)
real roots (1)

5a) $y=3x^2-4x+5$
$a=3,\ b=-4,\ c=5$
$\Delta=(-4)^2-4\times 3\times 5$
$=16-60$
$=-44$ (2)
does not cross X axis.

b) $kx^2-8x+k=0$
$a=k,\ b=-8,\ c=k$.
$\Delta=b^2-4ac$
$=(-8)^2-4\times k\times k$
$=64-4k^2$
For equal roots, $\Delta=0$
$64-4k^2=0$
$64=4k^2$
$k^2=16$ (2)
$k=\pm 4$.

c) $y=m(x-1)$
$y=2x^2$
$2x^2=m(x-1)$
$2x^2-mx+m=0$
$a=2,\ b=-m,\ c=m$
$\Delta=(-m)^2-4\times 2\times m$
$=m^2-8m$
Tangent, equal roots.
$\Delta=0$
$m^2-8m=0$
$m(m-8)=0$
$m=0$ or 8. (2)

d) $2x^2-x+3=0$
$a=2,\ b=-1,\ c=3$
i) $\alpha+\beta=-\frac{b}{a}$
$\therefore \alpha+\beta=\frac{1}{2}$ (1)
ii) $\alpha\beta=\frac{c}{a}$
$\alpha\beta=-\frac{3}{2}$ (1)
iii) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
$=(\frac{1}{2})^2-2\times\frac{3}{2}$
$=\frac{1}{4}+3$
$=3\frac{1}{4}=\frac{13}{4}$ (1)
iv) $\alpha^2\beta^2=(-\frac{3}{2})^2=\frac{9}{4}$ (1)
v) eqn is:
$x^2-(\alpha^2+\beta^2)x+\alpha^2\beta^2=0$
$x^2-\frac{13}{4}x+\frac{9}{4}=0$
$4x^2-13x+9=0$ (1)

e) $mx^2+x+n=0$
$2+-1=\frac{-b}{a}$
$1=-\frac{1}{m}$
$m=-1$
$2\times -1=\frac{c}{a}$
$-2=+\frac{n}{m}$
$-2=\frac{n}{-1}$
$n=2$
$\therefore m=-1,\ n=2$. (2)

f) $x^2+7x+j=0$
Roots $\alpha,\ \frac{1}{\alpha}$
Product $\alpha\times\frac{1}{\alpha}=\frac{c}{a}$
$1=\frac{j}{1}$
$\therefore j=1$ (1)

g) $kx^2-12x+3k$
$a=k,\ b=-12,\ c=3k$.
$\Delta=b^2-4ac$
$=(-12)^2-4k(3k)$
$=144-12k^2$
For unreal roots, $\Delta<0$
$144-12k^2<0$
$144<12k^2$
$k^2>12$
$\therefore k<-\sqrt{12}$ or $k>\sqrt{12}$.

For positive definite
$a>0$
so $k>0$. (2)
$\therefore k>\sqrt{12}$.