Yr 12 2005 June – SYD. GIRLS H.S.
Ext. 1 Task 3 . 75 min exam

QUESTION ONE

a) Evaluate $3\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (1 mark)

b) If $x^3+x^2-4x-1=0$ has roots α,β and γ. Find the value of (9 marks)

i) $\alpha+\beta+\gamma$
ii) $\alpha\beta+\beta\gamma+\alpha\gamma$
iii) $\alpha\beta\gamma$
iv) $\alpha^2+\beta^2+\gamma^2$
v) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$
vi) $(\alpha-1)(\beta-1)(\gamma-1)$

c)
i) Find the inverse function of $y=\ln x+5$ (1 mark)
ii) State the domain and the range of this inverse function. (2 marks)

d)
i) Solve $2x^3-x^2-2x+1=0$ (2 marks)
ii) Hence sketch $P(x)=2x^3-x^2-2x+1$ (2 marks)
iii) Hence solve $2x^3-x^2-2x+1>0$ (1 mark)

e)
i) Express $4\sin x-\cos x$ in the form $R\sin(x-\alpha)$, where R>0 and α is acute
ii) Hence solve $4\sin x-\cos x=1$, $0^\circ\le x\le 360^\circ$ (4 marks)

f) (3 marks)
i) Show that $P(x)=e^x+x-2$ has only one real root that lies between 0.4 and 0.5.
ii) By using one application of Newton's method and using 0.5 as the first approximation find this root correct to 2 decimal places.

QUESTION TWO

a) Show that $\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{3}{5}\right)=\frac{\pi}{4}$ (3 marks)

b) Given $f(x)=x^2+2x+4$
i) State the largest domain including $x=0$ for which $f(x)$ has an inverse function $f^{-1}(x)$. (1 mark)
ii) Find the equation of this inverse. (1 mark)

c) Use $u=1-2x$ to evaluate $\int_0^{\frac{1}{2}} x\sqrt{1-2x}\,dx$ (3 marks)

d) Find $\frac{dy}{dx}$ given (6 marks)
i) $y=\frac{5}{\cos^{-1}x}$
ii) $y=\sin^{-1}(e^x)$
iii) $y=(\tan^{-1}5x)^3$

e) Evaluate (3 marks)

$$\int_0^2 \frac{-dx}{\sqrt{4-x^2}}$$

f) Find the equation of the tangent to $y=\tan^{-1}x$ at the point where $x=1$. (3 marks)

g) Find $\int \sin^3 x\cos x\,dx$ using substitution $u=\sin x$ (2 marks)

h) The polynomial $(x-a)^3+b$ has a zero at $x=1$ and when is divided by x, the remainder is -7. Find all the values of a and b. (3 marks)

QUESTION THREE

a) If $x = \cos^{-1} t$ and $y = \sin^{-1} t$, find $\frac{dy}{dx}$ (3 marks)

b) Find the volume of the solid of revolution formed when the area under $y = \frac{1}{\sqrt{4+x^2}}$ is rotated about the x-axis between $x = 0$ and $x = 2$ (3 marks)

c) Given $y = 2\sin^{-1}(x-1)$

i) State the domain and the range of the function. (2 marks)
ii) Sketch the function (2 marks)

d) The roots of $x^3 - 9x^2 + 11x + 21 = 0$ are in arithmetic progression. Find them. (3 marks)

e) If $A(x) = x^4 - 4x^3 - x^2 + bx + a$ is divisible by $P(x) = x^2 - 4$, find the value of a and b. (3 marks)

f) Find $\frac{d}{dx}\left[\tan^{-1}\left(\frac{x+b}{a}\right)\right]$, hence find $\int \frac{dx}{a^2 + (x+b)^2}$ (4 marks)

g) Given $\tan^{-1} y = 2\tan^{-1} x$ (5 marks)

i) Express y as a function of x
ii) Show that the function has no turning point
iii) State the domain of the function
iv) Sketch the graph of the function.

END OF THE PAPER

Yr 12 Ext 1 2005

1\. a) $y = 3\sin^{-1}\frac{\sqrt{3}}{2}$

$\frac{y}{3} = \sin^{-1}\frac{\sqrt{3}}{2}$

$\sin\frac{y}{3} = \frac{\sqrt{3}}{2}$

$\frac{y}{3} = 60^\circ$

$y = 180^\circ$

b) i) $\alpha + \beta + \gamma = \frac{-b}{a}$

$= \frac{-1}{1}$

$= -1$

ii) $\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$

$= -4$

iii) $\alpha\beta\gamma = \frac{-d}{a}$

$= 1$

iv) $(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2\alpha\beta+2\beta\gamma+2\alpha\gamma$

$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$

$= (-1)^2 - 2(-4)$

$= 1 + 8$

$= 9$

v) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$

$= \frac{-4}{+1}$

$= -4$

vi) $(\alpha-1)(\beta-1)(\gamma-1) =$
$\alpha\beta\gamma - (\alpha\beta+\beta\gamma+\alpha\gamma)+\alpha+\beta+\gamma$.

$= 1-(-4)+(-1)-1$

$= 1+4-1+1$

$= 3$

c) i)

$x = \ln y + 5$

$x - 5 = \ln y$

$y = e^{x-5}$

ii) D: all real x
R: $y > 0$

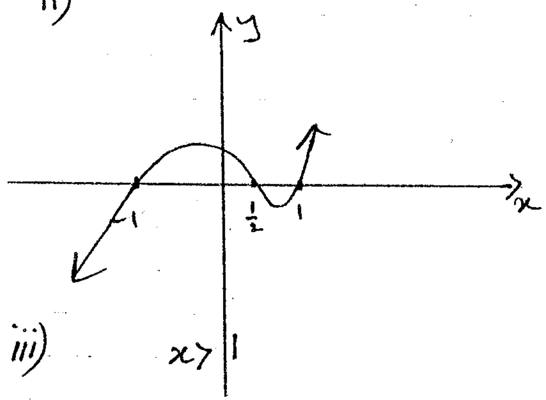
1 d) i)

$x^2(2x-1) - (2x-1) = 0$

$(2x-1)(x^2-1) = 0$

$x = \frac{1}{2}, 1$ or -1

ii)



iii) $x > 1$

$-1 < x < \frac{1}{2}, \; x > 1$

e)

i) $4\sin x - \cos x = \sqrt{17}\sin(x - 14^\circ 2')$

ii) $\sqrt{17}\sin(x - \tan^{-1}\frac{1}{4}) = 1$

$\sin(x - \tan^{-1}\frac{1}{4}) = \frac{1}{\sqrt{17}}$

$x - \tan^{-1}\frac{1}{4} = 14^\circ 2', 165^\circ 58'$

$x = 28^\circ 4, 180^\circ$

f) i) $y = e^x + x - 2$

$f(0.4) \doteqdot -0.108 < 0$

$f(0.5) \doteqdot 0.1487 > 0$

Since sign change $\therefore$ atleast one root between 0.4 & 0.5

ii)

$f'(x) = e^x + 1$

$x_2 = 0.5 - \frac{e^{0.5}+0.5-2}{e^{0.5}+1}$

$\doteqdot 0.44385$

$\therefore$ Root is 0.44

Q2

a) Let $x = \tan^{-1}\frac{1}{4}$

Let $y = \tan^{-1}\frac{3}{5}$

$\tan x = \frac{1}{4}, \; \tan y = \frac{3}{5}$

$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

$= \frac{\frac{1}{4}+\frac{3}{5}}{1-\frac{1}{4}\times\frac{3}{5}}$

$= \frac{\frac{17}{20}}{\frac{17}{20}}$

$= 1$

$x + y = \tan^{-1} 1$

$= \frac{\pi}{4}$

$\therefore \boxed{\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{3}{5} = \frac{\pi}{4}}$

2b) i) Domain: $x \geq -1$

ii) $y = x^2+2x+4$

$x = y^2+2y+4$

$x-4 = y^2+2y$

$x-4+1 = y^2+2y+1$

$(x-3) = (y+1)^2$

$y+1 = \sqrt{x-3}$

$y = -1+\sqrt{x-3}$

c)
$u = 1-2x$

$\frac{du}{dx} = -2$

$du = -2dx$

$\int_1^0 \frac{1-u}{2}\sqrt{u}\cdot\frac{-du}{2}$

$\frac{1}{4}\int_0^1 u^{\frac{1}{2}} - u^{\frac{3}{2}}\, du$

$= \frac{1}{4}\left[\frac{2u^{\frac{3}{2}}}{3} - \frac{2u^{\frac{5}{2}}}{5}\right]_0^1$

$= \frac{1}{4}\left[\frac{2}{3}-\frac{2}{5}\right] = \frac{1}{15}$

d) i) $y' = \frac{5}{\sqrt{1-x^2}\,(\cos^{-1}x)^2}$

ii) $y' = \frac{1}{\sqrt{1-e^{2x}}}\cdot e^x$

$= \frac{e^x}{\sqrt{1-e^{2x}}}$

iii) $y' = 3(\tan^{-1}5x)^2\cdot\frac{5}{1+25x^2}$

$= \frac{15(\tan^{-1}5x)^2}{1+25x^2}$

e)
$\left[\cos^{-1}\frac{x}{2}\right]_0^2$

$= \cos^{-1}1 - \cos^{-1}0$

$= 0 - \frac{\pi}{2}$

$= -\frac{\pi}{2}$

f) $y' = \frac{1}{1+x^2}$ at $x = 1$

$m = \frac{1}{2}$, $y = \frac{\pi}{4}$

$y - \frac{\pi}{4} = \frac{1}{2}(x-1)$

$\boxed{y = \frac{1}{2}x - \frac{1}{2} + \frac{\pi}{4}}$

2g) $\int \sin^3 x\cos x\,dx$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\,dx$

$\int u^3\,du$

$= \frac{u^4}{4} + c$

$= \frac{\sin^4 x}{4} + c$

h)
$P(1) = (1-a)^3 + b$

$0 = (1-a)^3 + b$

$\therefore$

$-b = (1-a)^3$ —— ①

$P(0) = (0-a)^3 + b$

$-7 = -a^3 + b$

$-b = -a^3 + 7$ —— ②

Equate ① & ②

$(1-a)^3 = -a^3+7$

$1 - 3a + 3a^2 - \cancel{a^3} = -\cancel{a^3} + 7$

$3a^2 - 3a - 6 = 0$

$a^2 - a - 2 = 0$

$(a-2)(a+1) = 0$

$a = 2$ or -1

$\boxed{a=2} \rightarrow -b = -(2)^3+7$

$-b = -8+7$

$\boxed{b = 1}$

$\boxed{a=-1} \rightarrow -b = -(-1)^3+7$

$= 1+7$

$\boxed{b = -8}$

3a) $\frac{dx}{dt} = -\frac{1}{\sqrt{1-t^2}}$

$\frac{dy}{dt} = \frac{1}{\sqrt{1-t^2}}$

$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$

$= \frac{1}{\sqrt{1-t^2}}\cdot\frac{-\sqrt{1-t^2}}{1}$

$= -1$

b) $V = \pi\int_0^2 \frac{1}{4+x^2}\,dx$

$= \pi\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2$

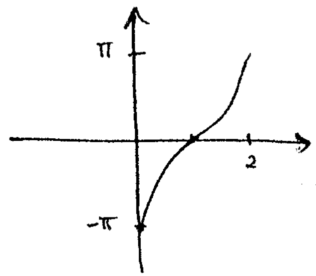
$= \pi\left[\frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0\right]$

$= \pi\left[\frac{1}{2}\cdot\frac{\pi}{4}\right] = \frac{\pi^2}{8}$

c) $y = 2\sin^{-1}(x-1)$

Domain: $0 \le x \le 2$

Range : $-\pi \le y \le \pi$



d) $a-d,\ a,\ a+d$

$a-d+a+a+d = +9$

$3a = 9$

$a = 3$

$(a-d)a(a+d) = -21$

$(3-d)\times 3(3+d) = -21$

$(9-3d)(3+d) = -21$

$27+9d-9d-3d^2 = -21$

$3d^2 = 48$

$d^2 = 16$

$d = \pm 4$

$a = 3 \quad a = 4$

$3-4,\ 3,\ 3+4$

$-1,\ 3,\ 7$

$\therefore$ roots are $-1, 3, 7$

e)

$A(2) = 16 - 32 - 4 + 2b + a$

$0 = -20 + 2b + a$

$\boxed{2b + a = 20}$ — (1)

$A(-2) = 16 + 32 - 4 - 2b + a$

$0 = 44 - 2b + a.$

$\boxed{2b - a = 44}$ — (2)

(2) + (1)

$4b = 64$

$\boxed{b = 16}$

$32 - a = 44$

$\boxed{a = -12}$

f) $\dfrac{d}{dx}\left(\tan^{-1}\dfrac{x}{a}\right) = \dfrac{a}{a^2+(x+b)^2}$

$$\int \frac{dx}{a^2+(x+b)^2}$$

$$= \frac{1}{a}\left(\tan^{-1}\frac{x+b}{a}\right) + c$$

g) i) $\tan^{-1} y = 2\tan^{-1} x$

take tan of both sides

$y = \tan(2\tan^{-1} x)$

let α be $\tan^{-1} x$

$y = \tan 2\alpha$

$y = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

$y = \dfrac{2x}{1-x^2}$

ii) $\dfrac{dy}{dx} = \dfrac{2(1-x^2)-(-2x)(2x)}{(1-x^2)^2}$

$= \dfrac{2-2x^2+4x^2}{(1-x^2)^2}$

$= \dfrac{2x^2+2}{(1-x^2)^2} > 0$

no t.pts

iii) Domain: all real x, $x \ne -1$ $x \ne 1$

iv)