SYDNEY GIRLS HIGH SCHOOL

2008 Assessment Task 3

MATHEMATICS EXTENSION 1

June 6th, 2008

Reading Time 5 minutes
Time allowed: 75 minutes

Topics: Integration by substitution, Inverse functions, Parametric equations of the parabola, Circle geometry, Polynomials.

INSTRUCTIONS:

- Attempt all FIVE questions.
- Questions are not of equal value.
- Start each question on a new page. Write on one side of the paper only.
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work.
- A table of standard integrals is provided on the back page of the examination.

TOTAL = 85 marks

Question 1

16 MARKS

a. The equation $x^3+2x-8=0$ has a root close to $x=1.6$. Use one application of Newton's method to find a further approximation to the root. Give your answer correct to 3 decimal places. *3 marks*

b. A polynomial is given by $P(x)=x^3+bx^2+cx-12$. Find the values of b and c if $(x-2)$ is a factor of $P(x)$ and 8 is the remainder when $P(x)$ is divided by $(x+2)$. *4 marks*

c. Find :

i. $\int\frac{dx}{3+4x^2}$ *2 marks*

ii. $\int x(2+x)^8\,dx$ using the substitution $x=u-2$. *4 marks*

d. Find the primitive of $\cos^2 x$. *3 marks*

Question 2

16 MARKS

a. Differentiate $\cos^{-1}(\sin x)$ and express the result in simplest form. *3 marks*

b. If α, β and γ are the roots of the equation

$5x^3 - 2x^2 + 4x - 1 = 0$, find the value of:

i. $\alpha + \beta + \gamma$ *1 mark*

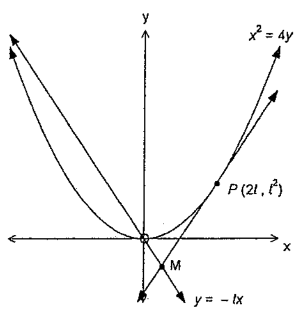
ii. $\alpha\beta\gamma$ *1 mark*

iii. $(2\alpha - 1)(2\beta - 1)(2\gamma - 1)$ *3 marks*

iv. $\alpha^2 + \beta^2 + \gamma^2$ *2 marks*

c.

$P(2t, t^2)$ is a point on the parabola $x^2 = 4y$. The tangent to the parabola at P and the line $y = -tx$ intersect at the point M.



i. Show that the tangent to the parabola has gradient t. *2 marks*

ii. Show that the equation of the tangent at P is given by the equation $tx - y - t^2 = 0$. *1 mark*

iii. Find the Cartesian equation of the locus of M as t varies. *3 marks*

Question 3

17 MARKS

a. Find the exact value of $\int_0^2 \frac{x}{\sqrt{1+x^2}}\,dx$, using the substitution $u = 1 + x^2$. *4 marks*

b. A polynomial is given by $P(x) = x^2(3-x)(2x+1)$.

i. Without differentiating, sketch the polynomial $P(x) = x^2(3-x)(2x+1)$. *3 marks*

ii. For what values of x is $P(x) > 0$? *2 marks*

iii. Write down the equation of the tangent to the curve $P(x)$ at the point where $x = 0$. *1 mark*

c.

i. Sketch the graph of the function $y = 3\sin^{-1}\left(\frac{x}{2}\right)$, clearly marking all key points. *3 marks*

ii. Find the exact equation of the normal to the curve $y = 3\sin^{-1}\left(\frac{x}{2}\right)$ at the point where $x = 1$. *4 marks*

Question 4

18 MARKS

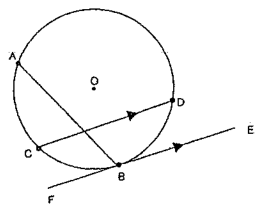
a.

i. Write down the expansion for $\tan(x-y)$. *1 mark*

ii. Hence evaluate $\tan^{-1}(5)-\tan^{-1}\left(\frac{2}{3}\right)$. *3 marks*

b. In the diagram, AB and CD are intersecting chords of a circle and CD is parallel to the tangent to the circle at B.

i. Copy the diagram onto your paper and prove that AB bisects $\angle CAD$, showing any extra construction lines.



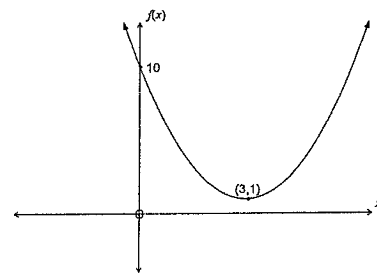
3 marks

ii. AB and CD intersect at P. Given that $CD = 13$ cm, $CP = 4$ cm and $AP = 12$ cm, calculate the length of AB. *2 marks*

Question 4 continued on the following page.

Question 4 *[continued]*

c. Below is the graph of $f(x)=1+(x-3)^2$.



i. State the largest positive domain for which $f(x)$ has an inverse function $f^{-1}(x)$. *1 mark*

ii. What is the equation of this inverse function $f^{-1}(x)$. *3 marks*

iii. Sketch $f^{-1}(x)$, clearly showing the domain and range. *2 marks*

iv. Find where the restricted function $f(x)$ and its inverse function $f^{-1}(x)$ intersect. *3 marks*

Question 5

18 MARKS

a. P $(2ap, ap^2)$ and Q $(2aq, aq^2)$ are points on the parabola $x^2 = 4ay$.

i. Show that the equation of the chord PQ is

$$y - \frac{1}{2}(p+q)x + apq = 0.$$

3 marks

ii. The chord PQ passes through $(0, 8a)$. Show that $pq = -8$.

1 mark

iii. Hence, if S is the focus of the parabola, show that

$$SP - SQ = a(p + \frac{8}{p})(p - \frac{8}{p}).$$

4 marks

b. For the parabola $x^2 = 12y$, the chord of contact from an external point $T(x_1, y_1)$ is perpendicular to TQ where Q is $(2x_1, 9y_1)$.

i. Write down the equation of the chord of contact from $T(x_1, y_1)$ to the parabola $x^2 = 12y$.

1 mark

ii. Hence, find the value of y_1.

3 marks

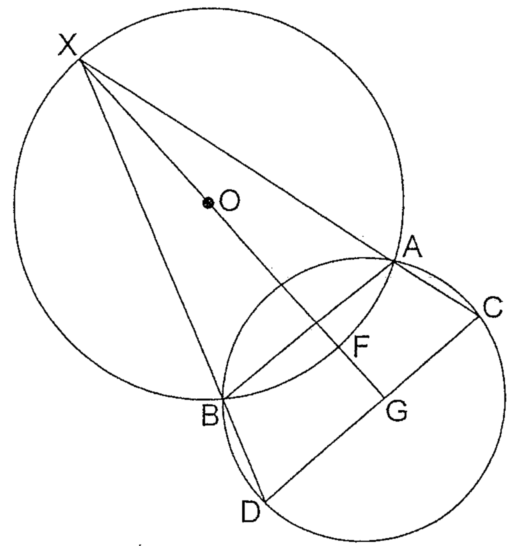
Question 5 continued on the following page.

Question 5 *[continued]*

c. Two circles cut at A and B. X is on the circle with centre O.

XA and XB cut the other circle at C and D respectively.

XO extended cuts the circle XAB at F and chord CD at G.



i. Neatly copy the diagram onto your answer sheet.

ii. Let $\angle AXF = x$. Prove that ACGF is a cyclic quadrilateral.

5 marks

iii. Prove that XG is perpendicular to CD.

1 mark

THE END

Year 12 – Mathematics Extension 1 Assessment Task 3
[Solutions]

TOTAL = /85

Question 1 = 16

a.
$f(x)=x^3+2x-8$
$f'(x)=3x^2+2$
$x_1=1.6$
$x_2=x_1-\dfrac{f(x_1)}{f'(x_1)}$
$x_2=1.6-\dfrac{(1.6)^3+2(1.6)-8}{3(1.6)^2+2}$ $=\dfrac{-0.704}{9.68}+1.6$ $=-0.073+1.6$
$\therefore x_2=1.673$
2 (3)

b.
$P(x)=x^3+bx^2+cx-12$
$P(2)=0 \quad , \quad P(-2)=8$
$\left.\begin{array}{l}(2)^3+b(2)^2+c(2)-12=0\\(-2)^3+b(-2)^2+c(-2)-12=8\end{array}\right\}$
$\left.\begin{array}{l}4b+2c-4=0\\4b-2c-28=0\end{array}\right\}$
$4c+24=0$
$4c=-24$
$\therefore c=-6$
$4b+2(-6)-4=0$
$4b=16$
$\therefore b=4 \quad , \quad c=-6$
3 (4)

c.i.
$\displaystyle\int\frac{dx}{3+4x^2}=\frac{1}{4}\int\frac{dx}{\frac{3}{4}+x^2}$
$=\dfrac{1}{4}\times\dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x}{\sqrt{3}}\right)+C$
$=\dfrac{1}{2\sqrt{3}}\tan^{-1}\left(\dfrac{2x}{\sqrt{3}}\right)+C$
1 (2)

c.ii.
$\displaystyle\int x(2+x)^8\,dx$
$x=u-2 \quad , \quad u=x+2$
$\dfrac{du}{dx}=1 \quad , \quad dx=du$
$\therefore I=\displaystyle\int(u-2)u^8\,du$
$=\displaystyle\int(u^9-2u^8)\,du$
$=\dfrac{u^{10}}{10}-\dfrac{2u^9}{9}+C$
$=\dfrac{(x+2)^{10}}{10}-\dfrac{2(x+2)^9}{9}+C$
3 (4)

Question 1 (continued)

d.
$\cos 2x=2\cos^2 x-1$
$\cos^2 x=\dfrac{1}{2}(\cos 2x+1)$
$\displaystyle\int(\cos^2 x)dx=\int\frac{1}{2}(\cos 2x+1)dx$
$=\dfrac{1}{2}\left(\dfrac{\sin 2x}{2}+x\right)+C$
$=\dfrac{\sin 2x}{4}+\dfrac{x}{2}+C$
(3)

Question 2 = 16

a.
$\dfrac{d}{dx}(\cos^{-1}(\sin x))=\dfrac{-1}{\sqrt{1-\sin^2 x}}\times\cos x$
$=\dfrac{-\cos x}{\sqrt{\cos^2 x}}$
$=\dfrac{-\cos x}{\cos x}$
$=-1$
(3) 2

b.i.
$5x^3-2x^2+4x-1=0$
$\alpha+\beta+\gamma=\dfrac{-b}{a}$
$=\dfrac{-(-2)}{5}$
$=\dfrac{2}{5}$
(1)

b.ii.
$\alpha\beta\gamma=\dfrac{-d}{a}$
$=\dfrac{-(-1)}{5}$
$=\dfrac{1}{5}$
(1)

Question 2 (continued)

b.iii.
$\alpha\beta+\alpha\gamma+\beta\gamma=\dfrac{c}{a}$
$=\dfrac{4}{5}$
$(2\alpha-1)(2\beta-1)(2\gamma-1)$
$=(2\alpha-1)(4\beta\gamma-2\beta-2\gamma+1)$
$=8\alpha\beta\gamma-4\alpha\beta-4\alpha\gamma+2\alpha-4\beta\gamma+2\beta+2\gamma-1$
$=8(\alpha\beta\gamma)-4(\alpha\beta+\alpha\gamma+\beta\gamma)+2(\alpha+\beta+\gamma)-1$
$=8\left(\dfrac{1}{5}\right)-4\left(\dfrac{4}{5}\right)+2\left(\dfrac{2}{5}\right)-1$
$=-1\frac{4}{5}$
1 (3)

b.iv.
$(\alpha+\beta+\gamma)^2=\alpha^2+\alpha\beta+\alpha\gamma+\alpha\beta+\beta^2+\beta\gamma+\alpha\gamma+\beta\gamma+\gamma^2$
$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$
$=\left(\dfrac{2}{5}\right)^2-2\left(\dfrac{4}{5}\right)$
$=-1\dfrac{11}{25}$
(2)

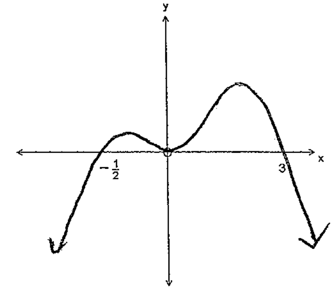
c.i.
$y=\dfrac{x^2}{4}$
$\dfrac{dy}{dx}=\dfrac{2x}{4} \quad , \quad \dfrac{dy}{dx}=\dfrac{x}{2}$
when $x=2t$, $\dfrac{dy}{dx}=\dfrac{2t}{2}$
$\therefore m_{\text{Tangent}}=t$
(2)

c.ii.
$y-t^2=t(x-2t)$
$y-t^2=tx-2t^2$
$\therefore tx-y-t^2=0.$
(1)

c.iii.
Intersection point of $y=-tx$ and $tx-y-t^2=0$
$tx-(-tx)-t^2=0$
$2tx=t^2$
$x=\dfrac{t}{2}$ Note: $\boxed{t=2x}$
$y=-t\left(\dfrac{t}{2}\right)$
$y=\dfrac{-t^2}{2}$ $\therefore M$ is $\left(\dfrac{t}{2},\dfrac{-t^2}{2}\right)$
$y=\dfrac{-(2x)^2}{2}$ i.e. $y=-2x^2$
2 (3)

Question 3 = 17

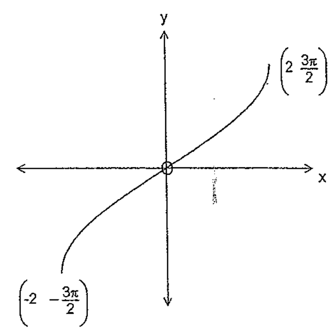
a.
$u=1+x^2 \quad , \quad \dfrac{du}{dx}=2x \quad , \quad 2x\,dx=du$
$\displaystyle\int_0^2\frac{x}{\sqrt{1+x^2}}\,dx=\frac{1}{2}\int_1^5\frac{du}{\sqrt{u}}$
$=\dfrac{1}{2}\displaystyle\int_1^5 u^{\frac{-1}{2}}\,du$
$=\dfrac{1}{2}\left[2u^{\frac{1}{2}}\right]_1^5$
$=\left[\sqrt{u}\right]_1^5$
$=\sqrt{5}-1$
(4)

b.i.


(3)

b.ii.
$\dfrac{-1}{2}<x<0$ and $0<x<3$
(2)

b.iii.
Equation of tangent at $x=0$ is $y=0$.
(1)

c.i.


(3)

Question 3 (continued)

c.ii.

$y = 3\sin^{-1}\left(\frac{x}{2}\right)$

$\frac{dy}{dx} = \frac{3}{\sqrt{4-x^2}}$

when $x = 1$, $m_T = \frac{3}{\sqrt{4-1}}$

and $y = 3\sin^{-1}\left(\frac{1}{2}\right)$ i.e. $y = \frac{\pi}{2}$

$\therefore m_N = \frac{-\sqrt{3}}{3}$

$y - \frac{\pi}{2} = \frac{-\sqrt{3}}{3}(x-1)$

$\therefore 2\sqrt{3}x + 6y = 3\pi + 2\sqrt{3}$

is the equation of the normal

Question 4

a.i.

$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

a.ii.

$\tan^{-1}(5) - \tan^{-1}\left(\frac{2}{3}\right)$

Let $x = \tan^{-1}(5)$ and $y = \tan^{-1}\left(\frac{2}{3}\right)$

$\therefore \tan x = 5$ and $\tan y = \frac{2}{3}$

$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

$= \frac{5 - \frac{2}{3}}{1 + 5 \times \frac{2}{3}}$

$= 1$

$\therefore x - y = \frac{\pi}{4}$ i.e. $\tan^{-1}(5) - \tan^{-1}\left(\frac{2}{3}\right) = \frac{\pi}{4}$

b.i.

Let $\angle CAB = x$

$\angle CAB = \angle CDB = x$ ($\angle$s in the same segment)

$\angle CDB = \angle DBE = x$ (alt. $\angle$s, $CD \parallel BE$)

$\angle DBE = \angle DAB = x$ ($\angle$ in alt. segment)

$\therefore \angle CAB = \angle DAB$ (both equal x)

$\therefore AB$ bisects $\angle CAD$

b.ii.

$AP \times BP = CP \times DP$

$12 \times BP = 4 \times 9$

$\therefore BP = 3$

$\therefore AB = 15$ cm

c.i.

$x \geq 3$

Question 4 (continued)

c.ii.

$y = 1 + (x-3)^2$

$f^{-1}(x)$ $x = 1 + (y-3)^2$

$(y-3)^2 = x - 1$

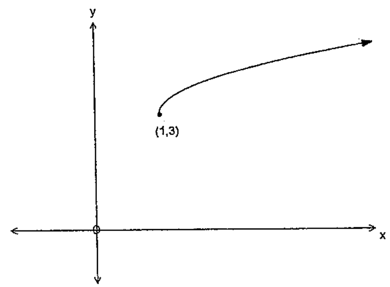
$y - 3 = \pm\sqrt{x-1}$

$y = 3 \pm \sqrt{x-1}$

The domain of the fn. is equivalent to the range of the inverse fn.

$\therefore y \geq 3$ $\therefore f^{-1}(x) = 3 + \sqrt{x-1}$

c.iii.



c.iv.

$f(x)$ and $f^{-1}(x)$ intersect on the line $y = x$

$\therefore$ Solve $f(x) = x$ or $f^{-1}(x) = x$.

$1 + (x-3)^2 = x$

$x^2 - 7x + 10 = 0$

$(x-2)(x-5) = 0$

$\therefore x = 2$ or $x = 5$

But the restricted function $f(x)$ is defined for $x \geq 3$.

$\therefore$ The only intersection point is $(5,5)$.

Question 5

a.i.

$m = \frac{aq^2 - ap^2}{2aq - 2ap}$

$= \frac{a(q^2 - p^2)}{2a(q-p)}$

$= \frac{(q-p)(q+p)}{2(q-p)}$

$= \frac{p+q}{2}$

$y - ap^2 = \frac{p+q}{2}(x - 2ap)$

$y - ap^2 = \frac{1}{2}(p+q)x - ap(p+q)$

$y - ap^2 = \frac{1}{2}(p+q)x - ap^2 - apq$

$\therefore y - \frac{1}{2}(p+q)x + apq = 0$

a.ii.

If chord passes through $(0, 8a)$

then it satisfies the equation of the chord

$\therefore 8a - \frac{1}{2}(p+q) \times 0 + apq = 0$

$8a + apq = 0$

$apq = -8a$

$\therefore pq = -8$

a.iii.

SP = distance from P to focus

= distance from P to directrix

(as per definition of the parabola)

$= ap^2 + a$ {see the diagram}

Similarly $SQ = aq^2 + a$

$\therefore SP - SQ = ap^2 + a - (aq^2 + a)$

$= ap^2 + a - aq^2 - a$

$= ap^2 - aq^2$

Since $pq = -8$, $q = \frac{-8}{p}$

$\therefore SP - SQ = ap^2 - a\left(\frac{-8}{p}\right)^2$

$= ap^2 - \frac{64a}{p^2}$

$= a\left(p^2 - \frac{64}{p^2}\right)$

$= a\left(p + \frac{8}{p}\right)\left(p - \frac{8}{p}\right)$

b.i.

$xx_1 = 2a(y + y_1)$

$a = 3$

$\therefore xx_1 = 6(y + y_1)$

b.ii.

$xx_1 = 6(y + y_1)$

$y = \frac{xx_1}{6} - y_1$, $\therefore m_{\text{chord of contact}} = \frac{x_1}{6}$

$m_{TQ} = \frac{9y_1 - y_1}{2x_1 - x_1}$

$= \frac{8y_1}{x_1}$

Since chord of contact and TQ are perp.

$m_{TQ} \times m_{\text{chord of contact}} = -1$

$\frac{8y_1}{x_1} \times \frac{x_1}{6} = -1$

$\therefore y_1 = \frac{-3}{4}$

c.ii.

Let $\angle AXF = x$

$\angle XAF = 90$ ($\angle$ in a semi-circle)

$\angle AFG = 90 + x$ (ext. $\angle$ of a Δ)

$\angle ABF = \angle AXF = x$ ($\angle$s in the same segment)

$\angle XBF = 90$ ($\angle$ in a semi-circle)

$\angle XBA = \angle XBF - \angle ABF$

$= 90 - x$

$\angle ACG = \angle XBA = 90 - x$ (ext. $\angle$ of cyclic quad.)

$\angle AFG + \angle ACG = (90 + x) + (90 - x)$

$= 180$

$\therefore ACGF$ is a cyclic quadrilateral.

c.iii.

$\angle FGC = \angle XAF = 90$ (ext. $\angle$ of cyclic quad.)

$\therefore XG$ is perpendicular to CD