SYDNEY GIRLS HIGH SCHOOL

YEAR 12 EXTENSION 1
MATHEMATICS

ASSESSMENT TASK 3

June 2010

Time allowed: 75 minutes
Plus 5 minutes reading time

Topics: **Induction, Integration by substitution, Polynomials, Inverse functions and Circle Geometry**

Instructions:

- There are Five (5) questions. Questions **are not** of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work.
- Start each question on a new page. Write on one side of the paper only.

Student Name: ... .

Teacher Name : . ...

QUESTION ONE (14 marks)

a) i) Show that the equation $x^3+x-3=0$ has a root between 1.2 and 1.3 (2)

ii) Taking 1.2 as the first approximation to the root, use Newton's method once to find a second approximation correct to 3 decimal places. (2)

b) Find $\int x\sqrt{16+x^2}\,dx$, using the substitution $u=16+x^2$ (3)

c) The roots of $x^3-x^2-5x+2=0$ are α, β and γ.
Write down the values of :

i) $\alpha\beta+\alpha\gamma+\beta\gamma$ (1)

ii) $\alpha\beta\gamma$ (1)

iii) Show that $\beta+\gamma=1-\alpha$ (1)

iv) Find the value of $\dfrac{\beta+\gamma}{\alpha}+\dfrac{\gamma+\alpha}{\beta}+\dfrac{\alpha+\beta}{\gamma}$ (2)

d) Find the general solution of $\sin\theta=\sqrt{3}\cos\theta$ (2)

QUESTION TWO (14 marks)

a) Find $\int \frac{dx}{1+4x^2}$ (2)

b) Given $f(x)=\log_e(x-3)$

i) find $f^{-1}(x)$ (2)

ii) State the domain and the range of $f^{-1}(x)$ (2)

c) i) State the domain and range of $y=3\cos^{-1}(x-2)$ (2)

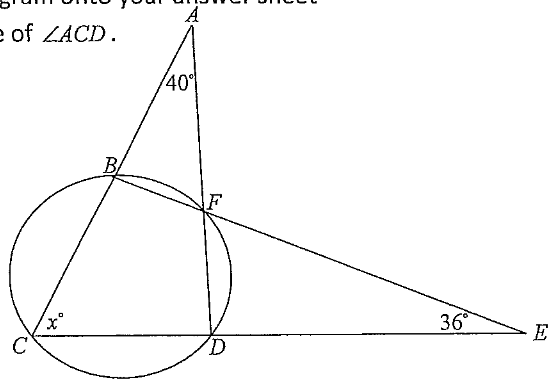
ii) Hence or otherwise sketch $y=3\cos^{-1}(x-2)$ (2)

d) Differentiate $y=\cos^{-1}\left(\frac{1}{x}\right)$ (2)

e)

i) Copy the diagram onto your answer sheet (2)

ii) Find the size of $\angle ACD$.



QUESTION THREE (14 marks)

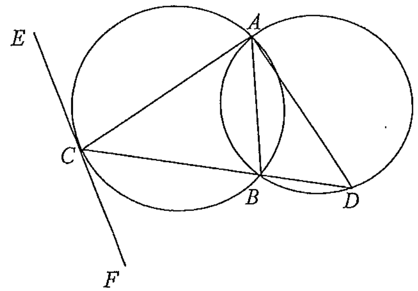
a) Evaluate $\int_0^4 \sqrt{16-x^2}\, dx$ using the substitution $x=4\sin\theta$. (3)

b) Differentiate $y=2\sin^{-1}4x+\sqrt{1-x^2}$ (3)

c) AC is tangent to the circle ABD and EF is a tangent to circle ABC.

i) Draw a neat sketch of the diagram on your answer sheet.

ii) Prove that $EF \parallel AD$ (2)



d) Use the method of mathematical induction to prove that $4\times 6^n+1$ is a multiple of 5, for all positive integers. (3)

e) The region bounded by the curve $y=\frac{1}{\sqrt{9+x^2}}$, the x-axis, $x=0$ and $x=\sqrt{3}$ is rotated about the x-axis. Find the volume of the solid of revolution. (3)

QUESTION FOUR (13 marks)

a) The polynomial $P(x)=ax^3+bx^2-8x+3$ has a zero at $x=1$ and has a remainder of 15 when divided by $(x+2)$.

i) Find the values of a and b (2)

ii) Hence factorise $P(x)$ fully (2)

iii) Draw a neat sketch of $y=P(x)$ (1)

iv) State the values of x for which $P(x)>0$ (1)

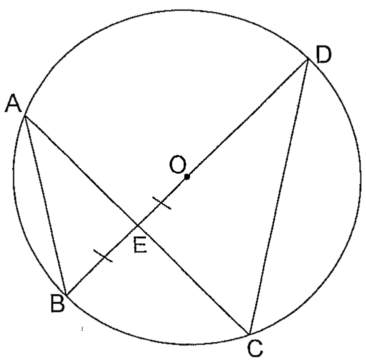
b) Evaluate $\int_1^e \frac{dx}{x(1+2\log_e x)^2}$ using the substitution $u=\log_e x$. (4)

c) The chord AC passes through the mid-point of OB. If $AE=12cm$ and $ED=16cm$,

i) Copy the diagram onto your answer sheet.

ii) Find the length of the diameter BD. (1)

iii) The length of EC. (2)



QUESTION FIVE (15 marks)

a) Differentiate $2x\sin^{-1}3x$ (3)

b) Prove that $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{4}$ (3)

c) i) Find $\frac{d}{dx}\left\{\frac{2x}{4+x^2}+\tan^{-1}\frac{x}{2}\right\}$ (3)

ii) hence, evaluate $\int_0^2 \frac{dx}{(4+x^2)^2}$ (3)

d) Solve $4x^3-12x^2+11x-3=0$, if the roots are in terms of an arithmetic series. (3)

Ex 1 2010 Assessment 3

1) a) i)

$f(1.2) = 1.2^3 + 1.2 - 3$

$= -\frac{9}{125}$

$f(1.3) = 0.497$

$f(1.2) < 0$ and $f(1.3) > 0$

$\therefore$ one root between $x = 1.2$ and $x = 1.3$

ii) $y = x^3 + x - x$

$y' = 3x^2 + 1$

$a_1 = 1.2 - \frac{f(1.2)}{f'(1.2)}$

$= 1.2 + \frac{9}{665}$

$= 1.214$

b) $u = 16 + x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{1}{2}\int 2x\sqrt{16+x^2}\,dx$

$= \frac{1}{2}\int \sqrt{u}\,du$

$= \frac{1}{2} \times \frac{2}{3} u^{\frac{3}{2}} + c$

$= \frac{1}{3}(16+x^2)\sqrt{16+x^2}$

c) i) $\alpha\beta + \beta\alpha + \alpha\gamma = \frac{c}{a}$

$= \frac{-5}{1}$

$= -5$

ii) $\alpha\beta\gamma = -\frac{d}{a}$

$= -2$

iii) $\alpha + \beta + \gamma = -\frac{b}{a}$

$= --1$

$= 1$

$\beta + \gamma = 1 - \alpha$

iv) $\frac{1-\alpha}{\alpha} + \frac{1-\beta}{\beta} + \frac{1-\gamma}{\gamma}$

$= \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} - 3$

$= \frac{\beta\gamma + \alpha\gamma + \beta\gamma - 3\alpha\beta\gamma}{\alpha\beta\gamma}$

$= \frac{-5 - 3(-2)}{-2}$

$= \frac{-5+6}{-2}$

$= -\frac{1}{2}$

d) $\frac{\sin\theta}{\cos\theta} = \sqrt{3}$

$\tan\theta = \sqrt{3}$

$\theta = \frac{\pi}{3}$

$\theta = n\pi + \tan^{-1}\sqrt{3} = n\pi + \frac{\pi}{3}$

Question 2 (14 marks)

a) $\int \frac{dx}{1+4x^2} = \int \frac{dx}{4(\frac{1}{4}+x^2)}$

$= \frac{1}{4}\int \frac{dx}{\frac{1}{4}+x^2}$

$= \frac{1}{4}\left[\frac{1}{\frac{1}{2}}\tan^{-1}\frac{x}{\frac{1}{2}}\right] + c$

$= \frac{1}{4}\left[2\tan^{-1}2x\right] + c$

(2) $= \frac{1}{2}\tan^{-1}2x + C$

b) $f(x) = \log_e(x-3)$

i) $y = \log_e(x-3)$

$x = \log_e(y-3)$

$e^x = y - 3$

$y = e^x + 3$ (2)

ii) D: all real x

R: $y > 3$ (2)

c) i) $y = 3\cos^{-1}(x-2)$

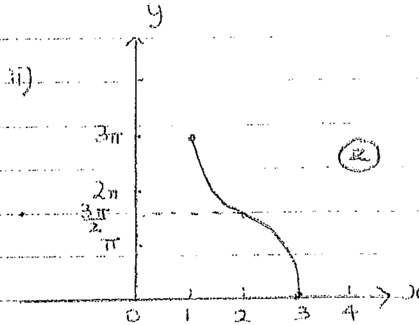
D: $-1 \le x-2 \le 1$

$1 \le x \le 3$ (1)

R: $0 \le \frac{y}{3} \le \pi$

$0 \le y \le 3\pi$ (1)

c) ii)



(2)

d) $y = \cos^{-1}\frac{1}{x}$

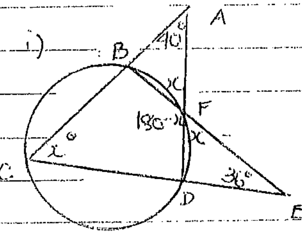
$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(\frac{1}{x})^2}} \times -x^{-2}$

$= \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}$

$= \frac{1}{x^2\sqrt{\frac{x^2-1}{x^2}}}$ (2)

$= \frac{1}{x\sqrt{x^2-1}}$

e) i)



(2)

ii) $\angle BFD = 180 - x$ (cyclic quad)

$\angle BFA = x$ (straight $\angle$)

$\angle EFD =$ (v. opposite)

$\angle FDE = 40 + x$ (exterior $\angle$ = sum of inter. opp)

$\therefore$ In ΔDFE

$40 + x + x + 36 = 180°$ ($\angle$ sum of Δ)

$76 + 2x = 180$

$2x = 104$

$x = 52°$

Q3)

$x = 4\sin\theta$

$\frac{dx}{d\theta} = 4\cos\theta$

$\int_0^{\frac{\pi}{2}} \sqrt{16 - 16\sin^2\theta} \, . \, 4\cos\theta \, d\theta$

$= \int \sqrt{16(1-\sin^2\theta)} \, . \, 4\cos\theta \, d\theta$

$= \int 4\cos\theta \, . \, 4\cos\theta \, d\theta$

$= \int 16\cos^2\theta \, d\theta$

$= 16 \times \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right] + c$

$= 8\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} + c$

$= 8\left[\frac{\pi}{2} + \frac{1}{2}\sin\frac{2\pi}{2} - 0 - \frac{1}{2}\sin 0\right]$

$= 8\left[\frac{\pi}{2} + 0\right]$

$= \frac{8\pi}{2}$

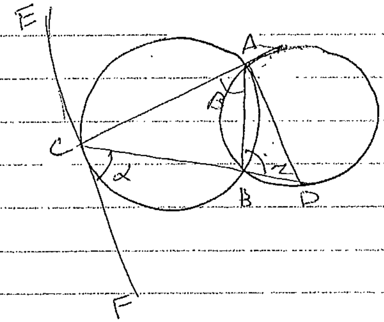
$= 4\pi$

b) $y' = 2 \times \frac{1}{\sqrt{1-(4x)^2}} \times 4$

$+ \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times (-2x)$

$= \frac{8}{\sqrt{1-16x^2}} - \frac{x}{\sqrt{1-x^2}}$

c)



$\alpha = B$ (Angle in alt segment)

$B = z$ (" " " ")

$\therefore \alpha = z$ Alt ∠'s are equal

$\therefore EF \parallel AD$

d) Prove true for n=1

$4 \times 6^1 + 1$

$= 25$

= divisible by 5

∴ true for n=1

Assume true for $n = k$

$$4 \times 6^k + 1 = 5p$$

Prove true for $n = k+1$

$$4 \times 6^{k+1} + 1 = 5q$$

$$LHS = 4 \times 6^k \times 6 + 1$$

$$= \cancel{4} \times \left(\frac{5p-1}{\cancel{4}}\right) \times 6 + 1$$

$$= 30p - 6 + 1$$

$$= 30p - 5$$

$$= 5(6p-1)$$

$$= 5q$$

$\therefore$ true for $n = k+1$
Since true for $n=1$
If true for $n=k$ $\therefore$ true for $n=2, 3$ and so on
$\therefore$ by Mathematical Induction true for all positive integers.

c.)
$$V = \pi \int_0^{\sqrt{3}} \frac{1}{9+x^2}\, dx$$

$$= \pi \left[\frac{1}{3} \tan^{-1} \frac{x}{3}\right]_0^{\sqrt{3}}$$

$$= \pi \left[\frac{1}{3} \tan^{-1} \frac{\sqrt{3}}{3} - \frac{1}{3} \tan^{-1} 0\right]$$

$$= \pi \left[\frac{1}{3} \times \frac{\pi}{6}\right]$$

$$= \frac{\pi^2}{18} u^3$$

4) $P(1) = a + b - 8 + 3$

$$\boxed{a + b - 5 = 0}$$

$$P(-2) = -8a + 4b + 16 + 3$$

$$-8a + 4b + 19 = 15$$

$$\boxed{-8a + 4b = -4}$$

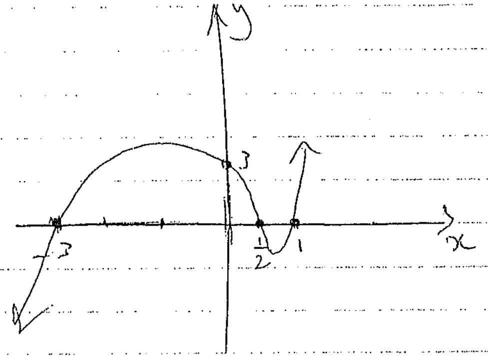
$$a = 5 - b$$
$$-8(5-b) + 4b = -4$$
$$-40 + 8b + 4b = -4$$
$$12b = 36$$
$$\boxed{b = 3}$$

$$a = 5 - 3$$
$$\boxed{a = 2}$$

ii)

$$\begin{array}{r|l} & 2x^2 + 5x - 3 \\ \hline x-1 & 2x^3 + 3x^2 - 8x + 3 \\ & 2x^3 - 2x^2 \\ \hline & 5x^2 - 8x \\ & 5x^2 - 5x \\ \hline & -3x + 3 \\ & -3x + 3 \end{array}$$

$$2x^3 + 3x^2 - 8x + 3 = (x-1)(2x^2 + 5x - 3)$$
$$= (x-1)(2x^2 + 6x - x - 3)$$
$$= (x-1)(2x(x+3) - (x+3))$$
$$= (x-1)(2x-1)(x+3)$$



iv) $-3 < x < \frac{1}{2}$ & $x > 1$

b) $u = \log_e x$

$\frac{du}{dx} = \frac{1}{x}$ $\therefore du = \frac{dx}{x}$

$\int \frac{du}{(1+2u)^2}$

$= (1+2u)^{-2}\, du$

$= \frac{(1+2u)^{-1}}{-1 \times 2}$

$= \left[\frac{-1}{2(1+2\log_e x)}\right]_1^e$

$= \frac{-1}{2(1+2\log_e e)} + \frac{1}{2(1+2\log_e 1)} = \frac{-1}{6} + \frac{1}{2}$

$= \frac{1}{3}$

5a) $y' = 2\sin^{-1} 3x + \frac{1}{\sqrt{1-(9x^2)}} \times 3 \times 2x$

$= 2\sin^{-1} 3x + \frac{6x}{\sqrt{1-9x^2}}$

b) Let $\alpha = \tan^{-1}\frac{1}{2}$ $\therefore \tan\alpha = \frac{1}{2}$

Let $\beta = \tan^{-1}\frac{1}{3}$ $\tan\beta = \frac{1}{3}$

$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$

$= \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}$

$= \frac{\frac{5}{6}}{1-\frac{1}{6}}$

$= \frac{\frac{5}{6}}{\frac{5}{6}}$

$= 1$

$\alpha+\beta = \tan^{-1} 1$

$= \frac{\pi}{4}$

c) i) $y' = \dfrac{2(4+x^2) - 2x(2x)}{(4+x^2)^2} + \dfrac{2}{4+x^2}$

$= \dfrac{8 + 2x^2 - 4x^2}{(4+x^2)^2} + \dfrac{2}{4+x^2}$

$= \dfrac{8 - 2x^2}{(4+x^2)^2} + \dfrac{2}{4+x^2}$

$= \dfrac{8 - 2x^2 + 2(4+x^2)}{(4+x^2)^2}$

$= \dfrac{8 - 2x^2 + 8 + 2x^2}{(4+x^2)^2}$

$= \dfrac{16}{(4+x^2)^2}$

$\displaystyle\int \frac{16}{(4+x^2)^2}\,dx = \frac{2x}{4+x^2} + \tan^{-1}\frac{x}{2}$

$\displaystyle\int_0^2 \frac{1}{(4+x^2)^2}\,dx = \frac{1}{16}\left[\frac{2x}{4+x^2} + \tan^{-1}\frac{x}{2}\right]_0^2$

$= \dfrac{1}{16}\left[\dfrac{4}{4+4} + \tan^{-1}1 - \left(0 + \tan^{-1}0\right)\right]$

$= \dfrac{1}{16}\left(\dfrac{4}{8} + \dfrac{\pi}{4}\right) = \dfrac{1}{32} + \dfrac{\pi}{64}$