SYDNEY GIRLS HIGH SCHOOL

2004 HSC Assessment Task 3

June 3, 2005

MATHEMATICS Extension 2

Year 12

Time allowed: 90 minutes

Topics: Polynomials, Integration, Volumes

DIRECTIONS TO CANDIDATES:

- Attempt all questions
- Questions are not of equal value
- There are 16 questions
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work
- Board approved calculators may be used
- Write on one side of the paper only

1) Solve the equation $4x^3-24x^2+23x+18=0$ given the roots are in arithmetic progression. [6]

2) If $P(x) = x^4 - ax + b$ has a double zero, show that $27a^4 = 256b^3$ [6]

3) A polynomial $P(x) = 3x^3 - 6x^2 + 9x - 1$ has roots α, β and γ. Find

a) $\alpha^3+\beta^3+\gamma^3$

b) $\alpha^4+\beta^4+\gamma^4$

c) the polynomial equation with roots α^2, β^2 and γ^2

d) the polynomial equation with roots α^2-1, β^2-1 and γ^2-1 [8]

4) If the polynomial $P(x)=x^4-4x^3+11x^2-14x+10$ has two roots of the form $a+ib$ and $a-2ib$ where a and b are real, find the 4 roots of the polynomial. [6]

5) The area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab.

The base of a solid is a circle of radius 4 cm. Each cross section perpendicular to a diameter is half of an ellipse where the major axis (a) is twice the minor axis(b). Find the volume of the solid. [6]

6) The circle $(x-3)^2+(y+2)^2=1$ is rotated about the X axis. Find the volume so formed. [6]

7) The area in the first quadrant enclosed by the curve $y=4x-x^3$ and the X axis is rotated about the Y axis. Find the volume that is formed. [6]

8) Find the integral: $\int\frac{dx}{\sqrt{9-4x^2}}$ [6]

9) Find the integral: $\int\frac{x^2dx}{\sqrt{4-x^2}}$ [6]

10) Find the integral: $\int\frac{dx}{1+\cos x+\sin x}$ [6]

11) Find the integral: $\int\frac{dx}{1+3\cos^2 x}$ [6]

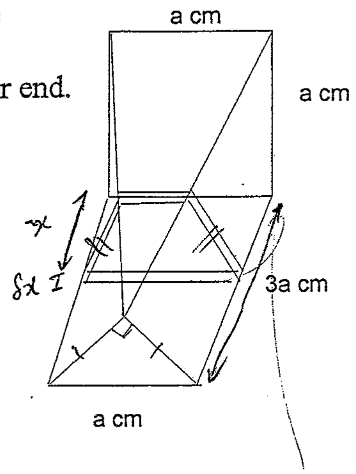
12) Find the integral: $\int\frac{x^2dx}{x^2+3x+2}$ [6]

13) Find the integral: $\int \frac{2x.dx}{\sqrt{x^2+2x+2}}$ [6]

14 A solid of length 3a cms has a right angled isosceles with hypotenuse of length a cm at one end and a square with each side length a cms at the other end.

Slices perpendicular to the axis of the solid are isosceles trapeziums.

a) Show that a slice of width ∂x taken x units from the triangular end is: $\left(\frac{x+3a}{12}\right)\left(a+\frac{x}{3}\right)\partial x \text{ cm}^3$



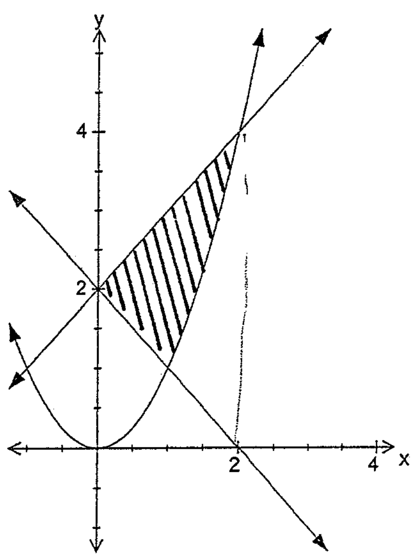
b) Hence find the volume of the solid

[6]

15 a) If $I_n = \int_0^{\frac{\pi}{2}} x^n.\sin x.dx$, show that $I_n = -n(n-1)\int_0^{\frac{\pi}{2}} x^{n-2}.\sin x.dx$

b) Hence evaluate $\int_0^{\frac{\pi}{2}} x^4.\sin x.dx$ [7]

16) The adjacent diagam shows the graphs of $y = 2\text{-}x$, $y = 2 + x$ and $y = x^2$.
The area enclosed by the graphs is to be rotated about the Y axis.
i) Find the area of the shape to be rotated
ii) Find the volume that is formed when it is rotated



[6]

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Q1. $4x^3 - 24x^2 + 23x + 18 = 0$

Roots are $\alpha, \alpha - d, \alpha + d$

Sum $= 3\alpha = \frac{24}{4} = 6$

$\therefore \alpha = 2$

$\therefore f(x) = (x-2)(4x^2 - 16x - 9) = 0$

$= (x-2)(2x-9)(2x+1)$

If $f(x) = 0$, $x = 2, 4\frac{1}{2}, -\frac{1}{2}$

Q2. $P(x) = x^4 - ax + b$

$P'(x) = 4x^3 - a$

If $P'(x) = 0$, $x^3 = \frac{a}{4}$, $x = \left(\frac{a}{4}\right)^{1/3}$

$P\left\{\left(\frac{a}{4}\right)^{1/3}\right\} = 0$

$\therefore \left(\frac{a}{4}\right)^{4/3} - a\left(\frac{a}{4}\right)^{1/3} + b = 0$

$\therefore \left(\frac{a}{4}\right)^{1/3}\left\{\frac{a}{4} - a\right\} = -b$

$\therefore \left(\frac{a}{4}\right)^{1/3}\left(+\frac{3a}{4}\right) = +b$

$\therefore \frac{a}{4} \cdot \frac{27a^3}{64} = b^3$

$\therefore 27a^4 = 256 b^3$

Q3. $3x^3 - 6x^2 + 9x - 1 = 0$

has roots α, β, γ

$\therefore 3\alpha^3 - 6\alpha^2 + 9\alpha - 1 = 0$

$3\beta^3 - 6\beta^2 + 9\beta - 1 = 0$

$3\gamma^3 - 6\gamma^2 + 9\gamma - 1 = 0$

$\therefore 3\Sigma\alpha^3 - 6\Sigma\alpha^2 + 9\Sigma\alpha - 3 = 0$

$\therefore \Sigma\alpha^3 = 2\Sigma\alpha^2 - 3\Sigma\alpha + 1$

$\Sigma\alpha = -b/a = 2$

$\Sigma\alpha\beta = c/a = 9/3 = 3$

$\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$= 4 - 6 = -2$

$\therefore \Sigma\alpha^3 = 2(-2) - 3(2) + 1$

$= -4 - 6 + 1$

$= -9$

$\Sigma\alpha^4$:

$\alpha . f(\alpha) = \beta . f(\beta) = \gamma . f(\gamma) = 0$

$\therefore 3\Sigma\alpha^4 - 6\Sigma\alpha^3 + 9\Sigma\alpha^2 - \Sigma\alpha = 0$

$\therefore 3\Sigma\alpha^4 = 6(-9) - 9(-2) + 2$

$= -54 + 18 + 2$

$= -34$

$\therefore \Sigma\alpha^4 = -\frac{34}{3}$

Roots $\alpha^2, \beta^2, \gamma^2$

Replace x by $\sqrt{x}$

$\therefore 3x\sqrt{x} - 6x + 9\sqrt{x} - 1 = 0$

$\therefore \sqrt{x}(3x + 9) = 6x + 1$

$\therefore x(9x^2 + 54x + 81) = 36x^2 + 12x + 1$

$\therefore 9x^3 + 18x^2 + 69x - 1 = 0$

Roots $\alpha^2 - 1, \beta^2 - 1, \gamma^2 - 1$

Replace x by $x + 1$ in last eqn.

$\therefore 9(x+1)^3 + 18(x+1)^2 + 69(x+1) - 1 = 0$

$\therefore 9(x^3 + 3x^2 + 3x + 1) + 18(x^2 + 2x + 1) + 69x + 68 = 0$

$\therefore 9x^3 + 45x^2 + 132x + 95 = 0$

Q4. $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$

has roots $a + ib$, $a - ib$, $a - 2ib$, $a + 2ib$

$\therefore P(x) = \{(x-a)^2 + b^2\}\{(x-a)^2 + 4b^2\}$

$\therefore P(x) = (x^2 - 2ax + a^2 + b^2)(x^2 - 2ax + a^2 + 4b^2)$

x^3 coeff: $-4 = -4a$, $\therefore a = 1$

x coeff: $-14 = -2a^3 - 8ab^2 - 2a^3 - 2ab^2$

but $a = 1$, $\therefore -14 = -4 - 10b^2$

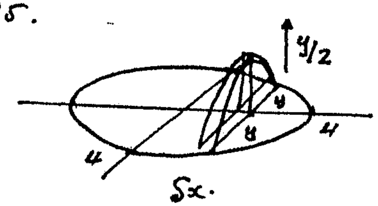
$\therefore 10b^2 = 10$

$b^2 = 1$, $b = \pm 1$

$\therefore$ the roots are

$1 \pm i$, $1 \pm 2i$

Q5.



$V_{slice} = \frac{1}{2}\pi ab\,\delta x$

$= \frac{\pi}{2} \cdot y \cdot y/2\, dx$

$= \frac{\pi}{4} y^2\, dx$

$= \frac{\pi}{4}(16 - x^2)\, dx$

$V_{solid} = \lim_{\delta x \to 0} \sum_{x=-4}^{4} \frac{\pi}{4}(16 - x^2)\, \delta x$

$= \frac{\pi}{2}\int_0^4 (16 - x^2)\, dx$

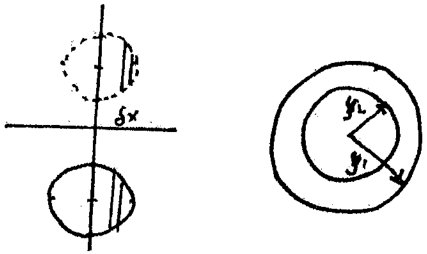
$= \frac{\pi}{2}\left[\frac{16x^2}{2} - \frac{x^3}{3}\right]$

$= \frac{\pi}{2}\left[64 - 64/3\right]$

$\therefore$ Volume $= 64\pi$

Q6. $(x-3)^2+(y+2)^2=1$

- move to $x^2+(y+2)^2=1$.



$\delta V=(\pi R^2-\pi r^2)\delta x$

$=\pi(y_1+y_2)(y_1-y_2)\,dx$

$(y+2)^2=1-x^2$

$y=-2\pm\sqrt{1-x^2}$

$y_1=|-2-\sqrt{1-x^2}|$, $y_2=|-2+\sqrt{1-x^2}|$

$\therefore y_1+y_2=4$, $y_1-y_2=2\sqrt{1-x^2}$

$\therefore \delta V=\pi.4.2\sqrt{1-x^2}\,dx$

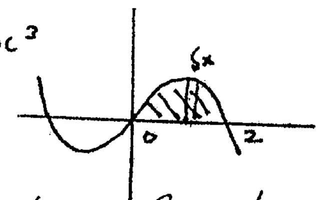
$V_{solid}=\lim\limits_{dx\to 0}\sum\limits_{x=-1}^{1}8\pi\sqrt{1-x^2}\,dx$

$=8\pi\int_{-1}^{1}\sqrt{1-x^2}\,dx$

$=8\pi\times\frac{1}{2}.\pi.1^2$ (semi-circle

$\therefore$ Volume $=4\pi^2\ u^3$

$\therefore\ y=4x-x^3$



By cylindrical shells,

Vol $=\pi R^2H-\pi r^2h$

$R=x+dx,\ r=x,\ H=h=y$

$\therefore \delta V=\pi y\{(x+dx)^2-x^2\}$

$=\pi y\{2x+dx\}dx$

$=2\pi yx\,dx,\quad dx^2\doteqdot 0$

$\therefore V_{solid}=\lim\limits_{dx\to 0}\sum\limits_{x=0}^{2}2\pi xy\,dx$

$=\int_0^2 2\pi x(4x-x^3)\,dx$

$=2\pi\int_0^2(4x^2-x^4)\,dx$

$=2\pi\left[\frac{4}{3}x^3-\frac{1}{5}x^5\right]_0^2$

$=2\pi\left[\frac{32}{3}-\frac{32}{5}\right]$

$=64\pi.\frac{2}{15}$

$\therefore$ Volume $=\frac{128\pi}{15}\ u^3$.

8. $\int\frac{dx}{\sqrt{9-4x^2}}=\frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right)+c.$

9. $\int\frac{x^2dx}{\sqrt{4-x^2}}$ Let $x=2\sin\theta$ $\therefore dx=2\cos\theta.d\theta.$

$=\int\frac{4\sin^2\theta.2\cos\theta.d\theta}{\sqrt{4-4\sin^2\theta}}$

$=\int\frac{4\sin^2\theta.2\cos\theta.d\theta}{2\cos\theta}$

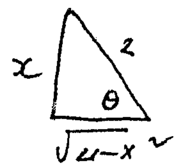
$=\int 4\sin^2\theta.d\theta$

$=2\int(1-\cos 2\theta)\,d\theta$

$=2\left[\theta-\frac{1}{2}\sin 2\theta\right]+c$

$=2\theta-\sin 2\theta+c$

$=2\theta-2\sin\theta.\cos\theta+c$



$=2\sin^{-1}\left(\frac{x}{2}\right)-\frac{2x\sqrt{4-x^2}}{4}+c.$

$=2\sin^{-1}\left(\frac{x}{2}\right)-\frac{x\sqrt{4-x^2}}{2}+c.$

10. $\int\frac{dx}{1+\cos x+\sin x}.$

Let $t=\tan x/2$

$\therefore \tan^{-1}t=x/2$

$\therefore \frac{1}{1+t^2}dt=\frac{1}{2}dx$

$+\ dx=\frac{2dt}{1+t^2},\ \sin x=\frac{2t}{1+t^2}$

$\cos x=\frac{1-t^2}{1+t^2}.$

$\therefore I=\int\frac{\frac{2dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}$

$=\int\frac{2dt}{1+t^2+1-t^2+2t}$

$=\int\frac{dt}{1+t}$

$=\ln(1+t)+c$

$=\ln(1+\tan x/2)+c.$

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11. $\int \frac{dx}{1+3\cos^2 x}$

$\div$ by $\cos^2 x$

$\therefore I = \int \frac{\sec^2 x}{\sec^2 x + 3}\, dx$

let $u = \tan x$ & $\sec^2 x = \tan^2 x + 1$

$\therefore du = \sec^2 x\, dx$

$\therefore I = \int \frac{du}{u^2+4}$

$= \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)$

$= \frac{1}{2}\tan^{-1}\left(\frac{\tan x}{2}\right) + c.$

12. $\int \frac{x^2\,dx}{x^2+3x+2} = \int \frac{x^2\,dx}{(x+1)(x+2)}$

Let $\frac{x^2}{(x+1)(x+2)} = A + \frac{B}{x+1} + \frac{C}{x+2}$

$\therefore x^2 = A(x+1)(x+2) + B(x+2) + C(x+1)$

equate x^2: $A = 1$

sub $x = -1$, $1 = B$

sub $x = -2$, $4 = -C$, $C = -4$

$\therefore I = \int \left(1 + \frac{1}{x+1} - \frac{4}{x+2}\right) dx$

$= x + \ln(x+1) - 4\ln(x+2) + c.$

13. $\int \frac{2x\,dx}{\sqrt{x^2+2x+2}}$

$= \int \frac{2x+2}{\sqrt{x^2+2x+2}}\,dx - 2\int \frac{dx}{\sqrt{(x+1)^2+1}}$

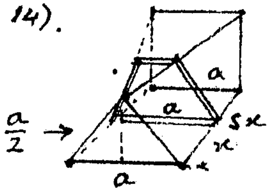
let $u = x^2+2x+2$

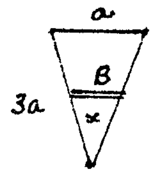
$\therefore du = (2x+2)\,dx$

$\therefore I_3 = \int \frac{du}{\sqrt{u}} - 2\int \frac{dx}{\sqrt{(x+1)^2+1}}$

$= 2\sqrt{x^2+2x+2} - 2\ln\left(x+1+\sqrt{x^2+2x+2}\right) + c$

14).



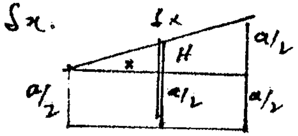


$\frac{x}{3a} = \frac{B}{a}$

$\therefore B = x/3$

$V_{slice} = \frac{h}{2}\{A+B\}\,\delta x.$

$A = a$, $B = \frac{x}{3}$



$h = a/2 + H,$

$\frac{H}{x} = \frac{a/2}{3a}$. $\therefore H = x/6$

& $h = \frac{a}{2} + \frac{x}{6} = \frac{x+3a}{6}$

$\therefore V_{slice} = \frac{x+3a}{12}\left(a + x/3\right)\delta x$

$\therefore V_{solid} = \lim_{\delta x \to 0} \sum_{x=0}^{3a} \left(\frac{x+3a}{12}\right)\left(a + x/3\right)\delta x$

$= \frac{1}{12}\int_0^{3a} \left(ax + \frac{x^2}{3} + 3a^2 + ax\right) dx$

$= \frac{1}{12}\left[\frac{x^3}{9} + 3a^2x + ax^2\right]_0^{3a}$

$= \frac{1}{12}\left\{\frac{27a^3}{9} + 9a^3 + 9a^3\right\}$

$= \frac{21a^3}{12} = \frac{7a^3}{4}$ $\therefore Vol = \frac{7a^3}{4}\ cm^3$.

15. $\int_0^{\pi/2} x^n \sin x\, dx$

Let $u = x^n$ $\quad dv = \sin x\, dx$

$du = nx^{n-1}dx$ $\quad v = -\cos x.$

$\therefore I = \left[-x^n\cos x\right]_0^{\pi/2} + n\int_0^{\pi/2} x^{n-1}\cos x\, dx$

$= n\int_0^{\pi/2} x^{n-1}\cos x.\, dx$

Let $u = x^{n-1}$ $\quad dv = \cos x\, dx$

$du = (n-1)x^{n-2}dx$ $\quad v = \sin x$

$\therefore I_n = n\left\{\left[x^{n-1}\sin x\right]_0^{\pi/2} - (n-1)\int_0^{\pi/2} x^{n-2}\sin x\right.$

$= n\left\{\left(\frac{\pi}{2}\right)^{n-1}\right\} - n(n-1)\int_0^{\pi/2} x^{n-2}\sin x.$

$\therefore \int_0^{\pi/2} x^4 \sin x\, dx$

$= 4\left(\frac{\pi}{2}\right)^3 - 12\int_0^{\pi/2} x^2\sin x\, dx$

& $\int_0^{\pi/2} x^2\sin x = 2\left(\frac{\pi}{2}\right) - 2\int_0^{\pi/2}(\sin x)\,dx.$

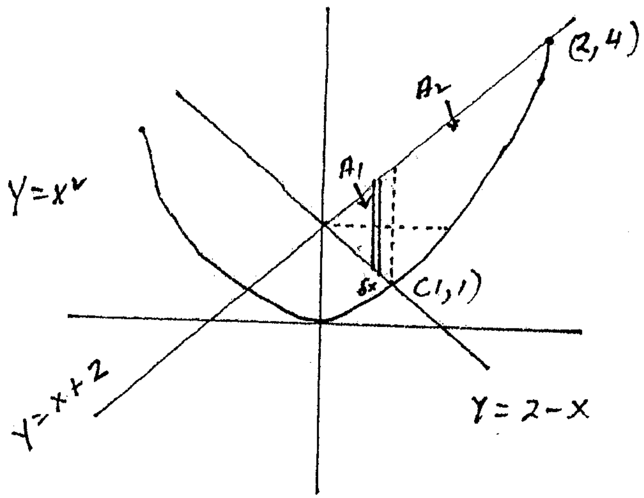
$= 2\left(\frac{\pi}{2}\right) + 2\left[\cos x\right]_0^{\pi/2}$

$= \pi - 2.$

$\therefore I_4 = \frac{\pi^3}{2} - 12(\pi - 2)$

$= \frac{\pi^3}{2} - 12\pi + 24.$

Q16.



Two areas to rotate.

1: between $y = 2+x$ & $y = 2-x$, $0 \le x < 1$

2: between $y = 2+x$ & $y = x^2$, $1 \le x \le 2$.

: $V_{shell} = \pi R^2 H - \pi r^2 h$

$R = x+\delta x$, $r = x$, $H = h = y_1 - y_2$
$= (x+2)-(2-x)$
$= 2x$.

$\delta V = \pi . 2x\{(x+\delta x)^2 - x^2\}$
$= 2\pi x\{2x+\delta x\}.\delta x$
$= 4\pi x^2 \delta x$, $\delta x^2 \doteqdot 0$

$V_{solid} = \lim\limits_{\delta x \to 0} \sum\limits_{x=0}^{1} 4\pi x^2 \delta x$
$= \int_0^1 4\pi x^2 dx$
$= \left[\frac{4}{3}\pi x^3\right]_0^1$
$= \frac{4\pi}{3}$.

A_2: $V_{shell} = \pi R^2 H - \pi r^2 h$
$R = x+\delta x$, $r = x$, $H = h = y_1 - y_2$
$= x+2-x^2$

$\therefore \delta V = \pi(x+2-x^2)\{(x+\delta x)^2 - x^2\}$
$= \pi(x+2-x^2)(2x+\delta x)\delta x$
$= \pi(2x^2+4x-2x^3)\delta x$, $\delta x^2 \doteqdot 0$

$\therefore V_{solid} = \lim\limits_{\delta x \to 0} \sum \pi(4x+2x^2-2x^3)\delta x$
$= \int_1^2 \pi(4x+2x^2-2x^3)dx$
$= 2\pi \int_1^2 (2x+x^2-x^3)dx$
$= 2\pi\left[x^2 + \frac{x^3}{3} - \frac{x^4}{4}\right]_1^2$
$= 2\pi\left\{\left(4+\frac{8}{3}-4\right) - \left(1+\frac{1}{3}-\frac{1}{4}\right)\right\}$
$= 2\pi\left\{\frac{8}{3} - \frac{13}{12}\right\}$
$= 2\pi\left\{\frac{32-13}{6\times 2}\right\}$
$= \frac{19\pi}{6}$

$\therefore$ Total volume
$= \frac{4\pi}{3} + \frac{19\pi}{6}$
$= \frac{27\pi}{6}$
$= \frac{9\pi}{2}\ u^3$.

Area:

$A_1 = \int_0^1 (x+2)-(2-x)\,dx$
$= \int_0^1 2x\,dx$
$= \left[x^2\right]_0^1$
$= 1$

$A_2 = \int_1^2 (x+2-x^2)\,dx$
$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_1^2$
$= (2+4-8/3) - (1/2+2-1/3)$
$= 6 - 8/3 - 1/2 - 2 + 1/3$
$= 4 - 7/3 - 1/2$
$= 7/6$

$\therefore$ Area $= \frac{13}{6}\ u^2$.