Sydney Girls High School

LABOR OMNIA VINCIT

# 2010 Assessment Task 3

# MATHEMATICS

## Extension Two

## Year 12

**Time allowed - 90 minutes (plus 5 minutes reading time)**

**Topics: Polynomials and Integration**

Instructions

- There are three questions worth 30 marks each
- All necessary working should be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- Write on one side of the paper only.
- A table of standard integrals is provided

**Name**__________ **Teacher**__________

## Question One (30 marks)

a) Find $\int \frac{x^3 dx}{x^4+1}$ [2]

b) Evaluate $\int_0^{\frac{\pi}{2}} \sin^3 x \cos x.dx$ [4]

c) Evaluate $\int_0^{\pi} \sin^3 x.dx$ [4]

d) Evaluate $\int_0^{\sqrt{7}} \frac{xdx}{\sqrt{x^2+9}}$ [4]

e) Evaluate $\int_1^4 \frac{dx}{x^2-2x+10}$ [4]

f) By completing the square find $\int \frac{1}{\sqrt{x^2-4x-5}}dx$ [3]

g) Find $\int xe^x dx$ [3]

h) i) Find the real numbers $A$, $B$ and $C$ [3]

such that $\frac{3x^2-3x+5}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$

ii) Hence find $\int \frac{3x^2-3x+5}{(x-1)(x^2+4)}dx$ [3]

## Question Two (30 marks)

a) Factorise $P(x)=x^3-2x^2+3x-6$ over the complex field [4]

b) The polynomial equation $x^3-2x^2+4x-8=0$ has roots $\alpha,\beta$ and $\gamma$. Find the value of:

i) $\alpha^{-1}+\beta^{-1}+\gamma^{-1}$ [2]

ii) $\alpha^2+\beta^2+\gamma^2$ [2]

iii) $\alpha^3+\beta^3+\gamma^3$ [3]

c) Solve the equation $4x^3-12x^2+3x+5=0$ given that the roots are in arithmetic progression [4]

d) The polynomial equation $x^3-2x^2+x-4=0$ has roots $\alpha,\beta$ and $\gamma$. Find the equation with roots:

i) $(\alpha+1),(\beta+1),(\gamma+1)$ [3]

ii) $\frac{1}{\alpha^2},\frac{1}{\beta^2},\frac{1}{\gamma^2}$ [4]

e) A polynomial $P(x)$ is odd. It has a double root at $x=1$ and passes through the point with co ordinates (2, 36). Find the equation of $P(x)$ [2]

f) i) Find the values of $a$ and $b$ ($a$ and $b$ both real) for which $(1-i)$ is a root of the equation $z^3+az+b=0$ [5]

ii) Hence find the remainder when $z^3+az+b=0$ is divided by $z-i$ [1]

## Question Three (30 marks)

a) Evaluate $\int_{-\pi}^{\pi}\sin x\cos x.dx$ [1]

b) Evaluate $\int_{-3}^{3}3\sqrt{9-x^2}dx$ [2]

c) Show that $\int\sec x\tan x.dx=\sec x+C$ [2]

d) Find $\int\frac{x^2+x}{x^2-x+1}dx$ [3]

e) i) If $\alpha$ is a multiple root of the polynomial $P(x)=0$ prove that $P'(\alpha)=0$ [2]

ii) Find all the roots of the equation $18x^3-15x^2-4x+4=0$ given that two of the roots are equal. [4]

f) Evaluate $\int_0^{\frac{\pi}{2}}\frac{dx}{3\sin x+4\cos x+5}$ using the substitution $t=\tan\frac{x}{2}$ [6]

g) Use integration by parts to find a reduction formula for $I_n=\int\frac{x^n}{\sqrt{1+x}}dx$ in terms of $I_{n-1}$ [5]

h) One root of the equation $x^3+px^2+qx+r=0$ is equal to the sum of the other two roots. Show that $p^3-4pq+8r=0$ [5]

3f) $I_n = \int \frac{x^n}{\sqrt{1+x}}\,dx$

let $u = x^n$ , $v' = (1+x)^{-\frac{1}{2}}$

$u' = nx^{n-1}$ $\quad v = 2(1+x)^{\frac{1}{2}}$

$$I_n = uv - \int v u'\,dx$$

$$= 2x^n\sqrt{1+x} - 2n\int x^{n-1}\sqrt{1+x}\,dx$$

$$= 2x^n\sqrt{1+x} - 2n\int x^{n-1}\sqrt{1+x} \times \frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx$$

$$= 2x^n\sqrt{1+x} - 2n\int \frac{x^{n-1}(1+x)}{\sqrt{1+x}}\,dx$$

$$= 2x^n\sqrt{1+x} - 2n\int\left(\frac{x^{n-1}}{\sqrt{1+x}} + \frac{x^n}{\sqrt{1+x}}\right)dx$$

$$= 2x^n\sqrt{1+x} - 2n\int\frac{x^{n-1}}{\sqrt{1+x}}\,dx - 2n\int\frac{x^n}{\sqrt{1+x}}\,dx$$

$$= 2x^n\sqrt{1+x} - 2n\,I_{n-1} - 2n\,I_n$$

$$(2n+1)I_n = 2x^n\sqrt{1+x} - 2n\,I_{n-1}$$

$$I_n = \frac{2x^n\sqrt{1+x}}{2n+1} - \frac{2n\,I_{n-1}}{2n+1}$$

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3g) let roots be $\alpha, \beta, \gamma$

and $\gamma = \alpha + \beta$

Sum of roots

$\alpha + \beta + \gamma = -p$

or $2\gamma = -p$

$\gamma = -\frac{p}{2}$ Now substitute in original

OR

Product of roots two at a time

$\alpha\beta + \alpha\gamma + \beta\gamma = q$

$\alpha\beta + \gamma(\alpha+\beta) = q$

$\alpha\beta + \gamma^2 = q$

$\alpha\beta + \frac{p^2}{4} = q$

3g cont

$\therefore \alpha\beta + \frac{p^2}{4} = q$

$\alpha\beta = q - \frac{p^2}{4}$

Now product of roots

$\alpha\beta\gamma = -r$

$\left(q - \frac{p^2}{4}\right)\left(-\frac{p}{2}\right) = -r$

$-\frac{pq}{2} + \frac{p^3}{8} = -r$

$-4pq + p^3 = -8r$

$p^3 - 4pq + 8r = 0$

5

Question 3

a) $\int_{-\pi}^{\pi} \sin x \cos x \, dx = 0$ since Fn odd ①

b) $3\int_{-3}^{3} \sqrt{9-x^2}\, dx = I$ using area of a semi circle radius 3

$I = 3 \times \frac{1}{2}(3)^2 \pi$

$= \frac{27\pi}{2}$ ②

c) $\frac{d}{dx}(\sec x) = \frac{d}{dx}(\cos x)^{-1}$

$= -1(\cos x)^{-2}(-\sin x)$

$= \frac{\sin x}{\cos^2 x}$

$= \sec x \tan x$

$\therefore \int \sec \tan x \, dx = \sec x + C$ ②

d) $\int \frac{x^2+x}{x^2-x+1} dx = I$

$\frac{x^2+x}{x^2-x+1} = \frac{x^2-x+1 \; +2x-1}{x^2-x+1}$

$= 1 + \frac{2x-1}{x^2-x+1}$

$I = \int 1 + \frac{2x-1}{x^2-x+1}$

$= x + \log_e |x^2-x+1| + C$ ③

e) $P(\alpha) = (x-\alpha)^n Q(x)$

$P'(\alpha) = n(x-\alpha)^{n-1} Q(x) + (x-\alpha)^n Q'(x)$

$= (x-\alpha)^{n-1}[nQ(x) + (x-\alpha)Q'(x)]$

put $x = \alpha$

$P'(x) = 0$ ②

Q3 e) ii) $18x^3 - 15x^2 - 4x + 4 = P(x)$

$P'(x) = 54x^2 - 30x - 4$

put $P'(x) = 0$ and divide by 2

$27x^2 - 15x - 2 = 0$

$(9x+1)(3x-2) = 0$

$x = -\frac{1}{9}$ or $x = \frac{2}{3}$.

Now $P(\frac{2}{3}) = 0$ ④

$\therefore P(x) = (3x-2)^2(2x+1)$

ie roots are $\frac{2}{3}, \frac{2}{3}, -\frac{1}{2}$

f) $I = \int_0^{\frac{\pi}{2}} \frac{dx}{3\sin x + 4\cos x + 5}$

let $t = \tan \frac{x}{2}$

when $x = \frac{\pi}{2}$, $t = 1$

$x = 0$, $t = 0$

$\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2}{1+t^2} dt$

$I = \int_0^1 \frac{\frac{2}{1+t^2}\, dt}{3\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right) + 5} \times \frac{1+t^2}{1+t^2}$

$= \int_0^1 \frac{2\, dt}{6t + 4 - 4t^2 + 5 + 5t^2}$

$= 2\int_0^1 \frac{dt}{t^2 + 6t + 9}$

$= 2\int_0^1 (t+3)^{-2}\, dt$

$= -2\left[(t+3)^{-1}\right]_0^1$

$= -2\left[\frac{1}{4} - \frac{1}{3}\right]$

$= \frac{1}{6}$ ⑥

Q2 c) $4x^3 - 12x^2 + 3x + 5 = 0$

Let roots be $\alpha - d$, $\alpha$, $\alpha + d$

Sum of roots $3\alpha = 3$

$\alpha = 1$

$x - 1$ is a factor

$$
\begin{array}{r|l}
 & 4x^2 \quad - 8x \quad - 5 \\
x-1 & 4x^3 - 12x^2 + 3x + 5 \\
 & 4x^3 \quad - 4x^2 \\
 & \quad -8x^2 \\
 & \quad -8x^2 \quad + 8x \\
 & \qquad -5x
\end{array}
$$

$(x-1)(4x^2 - 8x - 5) = 0$

$(x-1)(2x+1)(2x-5) = 0$

$x = 1, -\frac{1}{2}, 2\frac{1}{2}$

d) i) let $y = x + 1$ since $x = \alpha, \beta, \gamma$

subst $y - 1$ in place of $x$

$(y-1)^3 - 2(y-1)^2 + (y-1) - 4 = 0$

$y^3 - 3y^2 + 3y - 1 - 2y^2 + 4y - 2 + y - 1 - 4 = 0$

$y^3 - 5y^2 + 8y - 8 = 0$

ii) let $y = \frac{1}{x^2}$ since $x = \alpha, \beta, \gamma$

$\therefore x = \frac{1}{\sqrt{y}}$

$\left(\frac{1}{\sqrt{y}}\right)^3 - 2\left(\frac{1}{\sqrt{y}}\right)^2 + \frac{1}{\sqrt{y}} - 4 = 0$

$y^{-\frac{3}{2}} - 2y^{-1} + y^{-\frac{1}{2}} - 4 = 0$

$\times y^{\frac{3}{2}})$ $1 - 2y^{\frac{1}{2}} + y - 4y^{\frac{3}{2}} = 0$

$1 + y = 2y^{\frac{1}{2}}(2y + 1)$

square both sides

$1 + 2y + y^2 = 4y(4y^2 + 4y + 1)$

$1 + 2y + y^2 = 16y^3 + 16y^2 + 4y$

$0 = 16y^3 + 15y^2 + 2y - 1$

2e) $P(x) = Ax(x-1)^2(x+1)^2$

when $x = 2$, $P(x) = 36$

$36 = 2a(1)^2(3)^2$

$a = 2$

ie $P(x) = 2x(x-1)^2(x+1)^2$

f i) If $(1-i)$ is a root then $(1+i)$ is also

$\therefore (z-(1-i))(z-(1+i)) = z^2 - 2z + 2$ is a factor

$$
\begin{array}{r|l}
 & z \quad + 2 \\
z^2 - 2z + 2 & z^3 + 0 + az + b \\
 & z^3 - 2z^2 + 2z \\
 & \quad 2z^2 + (a-2)z + b \\
 & \quad 2z^2 \quad - 4z \quad + 4 \\
 & \quad a - 2 - (-4)z + b - 4 \\
 & \quad (a+2)z \quad + (b-4)
\end{array}
$$

Now remainder must be a factor

$\therefore a + 2 = 0$, $b - 4 = 0$

$a = -2$, $b = 4$

ie $z^3 - 2z + 4 = 0$

ii) remainder $= i^3 - 2i + 4$

$= 4 - 3i$

1a) $\frac{1}{4}\int \frac{4x^3\,dx}{x^4+1}$

$= \frac{1}{4}\log(x^4+1) + c$

b) let $u = \sin x$

$\frac{du}{dx} = \cos x$

$\frac{du}{\cos x} = dx$

when $x = \frac{\pi}{2}$, $u = \sin\frac{\pi}{2}$
$= 1$

" $x = 0$, $u = \sin 0$
$= 0$

$\int_0^1 u^3 \cancel{\cos x} \cdot \frac{du}{\cancel{\cos x}}$

$= \left[\frac{u^4}{4}\right]_0^1$

$= \frac{1}{4}$

c) $\int_0^\pi \sin x(1-\cos^2 x)\,dx$

$= \int_0^\pi \sin x\,dx - \int_0^\pi \sin x\cos^2 x\,dx$

$= \left[-\cos x\right]_0^\pi + \left[\frac{\cos^3 x}{3}\right]_0^\pi$

$= --1+1 + \frac{(-1)^3}{3} - \frac{1^3}{3}$

$= 1\frac{1}{3}$

d) let $u = x^2+9$

$\frac{du}{dx} = 2x$

$\frac{du}{2x} = dx$

when $\cancel{x} = \sqrt{7}$, $u = 7+9$
$= 16$

$x = 0$, $u = 0+9$
$= 9$

$\int_9^{16} \frac{x}{\sqrt{u}}\frac{du}{2x} = \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_9^{16}$
$= 4-3 = 1$

e) $\int_1^4 \frac{dx}{(x-1)^2+9}$

$= \frac{1}{3}\left[\tan^{-1}\frac{x-1}{3}\right]_1^4$

$= \frac{1}{3}(\tan^{-1}1 - \tan^{-1}0)$

$= \frac{1}{3}\times\frac{\pi}{4}$

$= \frac{\pi}{12}$

(f) $\int \frac{dx}{\sqrt{(x-2)^2-9}}$

$= \log\{x-2+\sqrt{x^2-4x+5}\}+c$

g) let $u = x$ $\quad v' = e^x$
$u' = 1$ $\quad v = e^x$

$\int xe^x\,dx = xe^x - \int e^x\,dx$

$= xe^x - e^x + c$

h) i) $3x^2-3x+5 = A(x^2+4)+(x-1)(Bx+C)$

when $x=1$, $5 = 5A$
$A = 1$

" $x = 0$, $5 = 4A - C$
$= 4 - C$
$\therefore C = -1$

$3x^2 = Ax^2 + Bx^2$

$3 = 1+B$

$\therefore B = 2$

ii) $\int\left(\frac{1}{x-1} + \frac{2x-1}{x^2+4}\right)dx$

$= \log(x-1) + \int\frac{2x}{x^2+4}\,dx - \int\frac{dx}{x^2+4}$

$= \log(x-1) + \log(x^2+4) - \frac{1}{2}\tan^{-1}\frac{x}{2} + c$

Question 2

a) $P(x) = x^3-2x^2+3x-6$

$P(1) = 1-2+3-6 \neq 0$

$P(-1) = -1-2-3-6 \neq 0$

$P(2) = 8-8+6-6 = 0$ $\quad\therefore$ $x-2$ is a factor of $P(x)$

$$\begin{array}{r l}
 & x^2 \quad\;\; + 3 \\
x-2 & \overline{)\,x^3 - 2x^2 + 3x - 6} \\
 & \underline{x^3 - 2x^2} \\
 & \qquad\qquad +3x-6 \\
 & \qquad\qquad \underline{3x-6} \\
 & \qquad\qquad\quad 0
\end{array}$$

$P(x) = (x-2)(x^2+3)$

$= (x-2)(x+\sqrt{3}i)(x-\sqrt{3}i)$

b) $x^3 - 2x^2 + 4x - 8 = 0$

i) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}$

$= \frac{4}{8}$

$= \frac{1}{2}$

ii) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= (-2)^2 - 2(4)$

$= -4$

iii) $x^3 = 2x^2 - 4x + 8 \cancel{=0}$

$\alpha^3 = 2\alpha^2 - 4\alpha + 8$

$\beta^3 = 2\beta^2 - 4\beta + 8$

$\gamma^3 = 2\gamma^2 - 4\gamma + 8$

$\alpha^3+\beta^3+\gamma^3 = 2(\alpha^2+\beta^2+\gamma^2) - 4(\alpha+\beta+\gamma) + 24$

$= 2(-4) - 4(2) + 24$

$= 8$