SYDNEY GIRLS HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

LABOR OMNIA VINCIT

1996

MATHEMATICS

3 UNIT (ADDITIONAL)
AND
3/4 UNIT (COMMON)

Time allowed - Two hours
(Plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES

Name [redacted]

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on the last page.
- Board-approved calculators may be used.
- Each question attempted should be started on a new sheet. Write on one side of the paper only.

This is a trial paper ONLY. It does not necessarily reflect the format or the contents of the 1996 HSC examination paper in this subject.

QUESTION 1 (start a new page)

a) Solve $\dfrac{2x+5}{x+1}<1$

b) Find the co ordinates of the point that divides the interval joining A(-1, 4) to B(7, 12) externally in the ratio 1 : 2

c) Differentiate with respect to x ;

i) $y=\sqrt{\sin x}$

ii) $y=\sin^{-1}(1-x)$

d) Evaluate $\int_{0.1}^{0.4}\sec^2 3x\ dx$ correct to 3dp

QUESTION 2 (start a new page)

a) Find the exact value of $\sin^{-1}\left(\dfrac{1}{2}\right)-\tan^{-1}\left(-\sqrt{3}\right)$

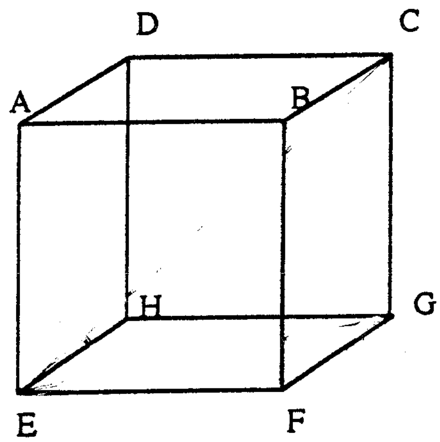
b) For the polynomial $P(x)=x^3+x-1$

i) show that a root exists between $x = 0$ and $x = 1$

ii) use one approximation of Newtons' method to achieve a better estimate of this root which lies near 0.5 correct to 2 decimal places.

c) Find the equation of the tangent to $y=\tan 3x$ at the point where $x=\dfrac{\pi}{3}$

d) The figure below is a cube.



Calculate the angle between CE and the plane EFGH (answer to the nearest minute).

QUESTION 3 (start a new page)

a) Given the function $y = 3\sin(2x + \pi)$;

i) state the period and amplitude

ii) sketch the graph for $0 \le x \le 2\pi$

b) Using the substitution $u = 1 + x^2$ find $\int x(1+x^2)^7 dx$

c) Use mathematical induction to show that $3^{2n} - 1$ is divisible by 8

QUESTION 4 (start a new page)

a) i) For what value of k is the polynomial $Q(x) = 4x^3 - x + k$ divisible by $2x + 3$?

ii) Use your answer from i) to fully factorise $Q(x)$

b) P is a point on the parabola $x^2 = 4y$. Show the normal to the curve at $P(2p, p^2)$ has equation $x = -py + 2p + p^3$

c) Given the function $6\cos^2\theta + 8\sin\theta\cos\theta$

i) Express the function in terms of $\cos 2\theta$ and $\sin 2\theta$

ii) Hence deduce an expression for the function in the form $A + 5\cos(2\theta - \alpha)$ where A and α are constants.

iii) Solve the equation $6\cos^2\theta + 8\sin\theta\cos\theta = 4$ for $0^\circ \le \theta \le 360^\circ$

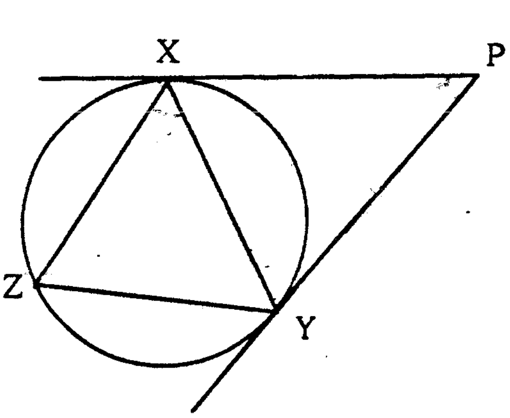
QUESTION 5 (start a new page)

a) i) Show that $P = P_0e^{kt}$ satisfies the equation $\frac{dP}{dt} = kP$

ii) In a culture of bacteria the number present P, is given by the formula $P = P_0e^{kt}$ where P_0 is the initial population of bacteria and k is a constant. If between 1am and 4am the population doubles, at what time would you expect the population to be ten times the 1am population ?

b) The velocity of a particle is given by $v = 2x + 1$ cms^{-1}. If the initial displacement is 1 cm to the right of the origin, find the displacement as a function of time.

c)

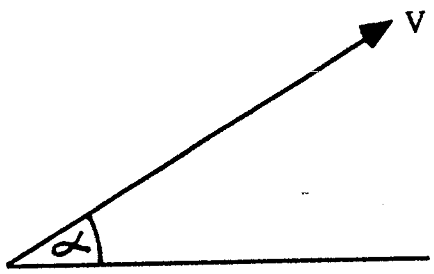


PX and PY are tangents, $\angle YXZ = \angle XPY = 2a^0$, prove XZ = XY giving reasons.

QUESTION 6 (start a new page)

a) Find the volume of the solid of revolution generated by rotating $y = \sin x$ around the X axis from $x = 0$ to $x = \pi$

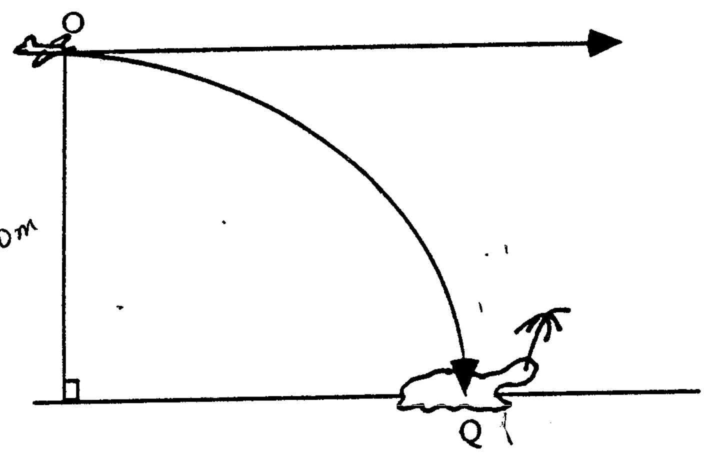
b)



i) A particle is projected with a velocity V ms^{-1} at an angle α to the horizontal. Show that the projectiles trajectory is defined by the equations

$$x = Vt\cos\alpha$$
$$y = -\frac{1}{2}gt^2 + Vt\sin\alpha$$

ii) A plane is flying horizontally at $400ms^{-1}$ at a height of 1280m above the ocean. It releases a survival package from a point O towards the centre of a small island Q



α) How far before Q should the package be released so that it falls on the centre of the island ? (use $g = 10ms^{-2}$)

β) Show that the speed of the package on impact is approximately $431ms^{-1}$

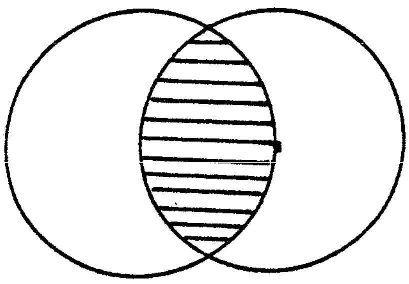
QUESTION 7 (start a new page)

a) i) Show that $\dfrac{5}{(x-2)(x+3)}$ can be expressed in the form $\dfrac{A}{x-2}+\dfrac{B}{x+3}$

ii) Hence or otherwise find $\displaystyle\int \frac{5\,dx}{(x-2)(x+3)}$

b) On a certain day in Fremantle Harbour the depth of high tide is 32 metres. At low tide $6\frac{1}{2}\,hrs$ later the depth of water is 21 metres. If ~~high~~ low. tide is 12.10am, what is the earliest time at which a ship needing 28.5 metres of water can enter the harbour. (Assume rise and fall of tide in SHM)

c)



Two equal circles of radius r are drawn passing through the centre of each other.

Show that the common area is $r^2\left(\dfrac{2\pi}{3}-\dfrac{\sqrt{3}}{2}\right)$ units^2

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Stephanie SIM.
8/7/02

Sydney Girls High School 1996 3/4 U.

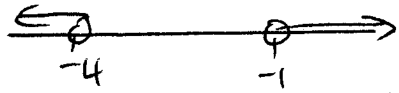
QUESTION 1

A.) $\frac{2x+5}{x+1} - 1 < 0$

$\frac{2x+5-x-1}{x+1} < 0$; $\frac{x+4}{x+1} < 0$

① $x+4 < 0$ $\quad$ $x+1 > 0$

$x < -4$ $\quad$ $x > -1$

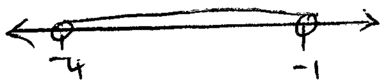


✓

$\therefore$ no solution

② $x+4 > 0$ $\quad$ $x+1 < 0$

$x > -4$ $\quad$ $x < -1$



$\therefore$ $\underline{\underline{-4 < x < 1}}$ ✓

B) Let point be $P(p, q)$

$P = \left(\frac{1(7) - 2(-1)}{1-2} , \frac{1(12) - 2(4)}{1-2} \right)$

$= \left(\frac{7+2}{-1} , \frac{12-8}{-1} \right)$

$= \underline{\underline{(-9, -4)}}$ ✓

c) i.) $y = \sqrt{\sin x} = (\sin x)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(\sin x)^{-\frac{1}{2}} \times \cos x$

$= \boxed{\frac{\cos x}{2\sqrt{\sin x}}} \times \frac{\sqrt{\sin x}}{\sqrt{\sin x}} = \frac{\sqrt{\sin x}\cos x}{2\sin x} = \frac{\sqrt{\sin x}\cot x}{2}$ ✓

ii.) $y = \sin^{-1}(1-x)$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-(1-x)^2}} \times -1$

$= \frac{-1}{\sqrt{1-(1-2x+x^2)}} = \frac{-1}{\sqrt{2x-x^2}}$ ✓

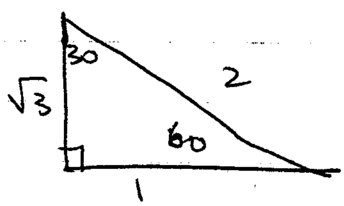
D) $\int_{0.1}^{0.4} \sec^2 3x \, dx$ RADIANS!

$= \frac{1}{3}(\tan 3x)_{0.1}^{0.4}$ ✓

$= \frac{1}{3}\{\tan 1.2 - \tan 0.3\}$

$= \frac{1}{3}\underline{2.263}$ (3dp) $=$ 0.754 rad (to 3 dp).

QUESTION 2



A.) $\sin^{-1}\left(\frac{1}{2}\right) - \tan^{-1}(-\sqrt{3})$

$= \sin^{-1}\left(\frac{1}{2}\right) + \tan^{-1}(\sqrt{3})$ ✓

$= \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi + 2\pi}{6} = \boxed{\frac{\pi}{2}}$ ✓

B) $P(x) = x^3 + x - 1$

i.) $P(0) < 0$

$P(1) > 0$ $\quad$ $P(x)$ is a continuous function ✓

$\therefore$ a root exists btw $x = 0$ and $x = 1$ ✓

ii.) $x_1 = 0.5$

$$x_2 = x_1 - \frac{P(x_1)}{P'(x_1)} \checkmark \qquad P'(x) = 3x^2 + 1$$

$$x_2 = 0.5 - \frac{P(0.5)}{P'(0.5)}$$

$= \underline{\underline{0.71}}$ (2dp) ✓

c) $y = \tan 3x$.

when $x = \frac{\pi}{3}$, $y = \tan \pi = \underline{0}$ ✓

$\therefore$ $(\frac{\pi}{3}, 0)$ lies on the curve $y = \tan 3x$.

$\frac{dy}{dx} = 3\sec^2 3x$. ✓

when $x = \frac{\pi}{3}$, $\frac{dy}{dx} = 3\sec^2 \pi = \underline{3}$ ✓

Eqt of tgt is $(y - 0) = 3(x - \frac{\pi}{3})$

$y - 3x + \pi = 0$

$\boxed{3x - y - \pi = 0}$ ✓

D) Let all sides of cube = x.
In Δ EFG,

$$(EG)^2 = (EF)^2 + (FG)^2$$
$$(EG)^2 = x^2 + x^2$$
$$(EG)^2 = 2x^2$$
$$\underline{EG = \sqrt{2}x} \checkmark$$ (EG > 0 because it is a length)

In Δ ECG, $\tan\angle CEG = \dfrac{CG}{EG}$

$$= \frac{x}{\sqrt{2}\cancel{x}} \checkmark$$

$$\tan\angle CEG = \frac{1}{\sqrt{2}}$$

S	A ✓
T ✓	C

$$\angle CEG = \underline{35^\circ 16'} \checkmark \text{ or } \underline{215^\circ 16'}$$

But angle is acute as shown in diagram.

$$\therefore \angle CEG = \underline{35^\circ 16'} \checkmark$$

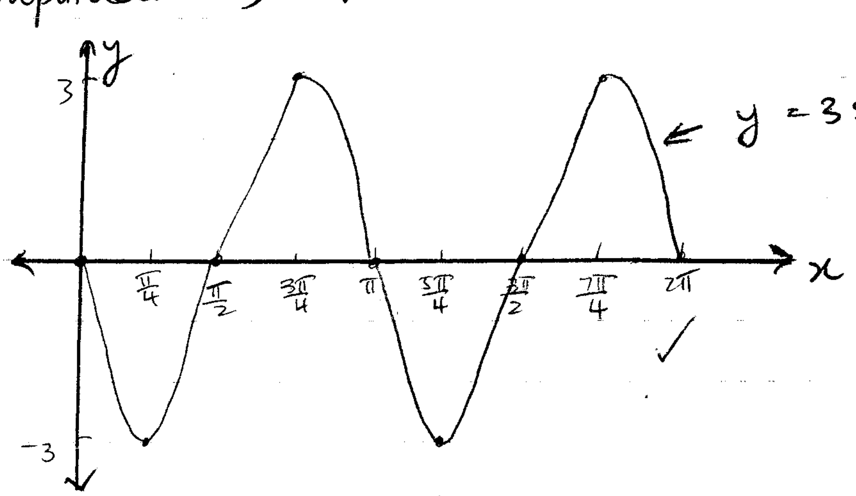
QUESTION 3

A.) $y = 3\sin(2x+\pi)$

i.) Period $= \frac{2\pi}{n} = \frac{2\pi}{2} = \underline{\pi}$ ✓

Amplitude $= 3$ ✓

ii.)



✓

B) $\int x(1+x^2)^7\,dx$

$u = 1+x^2$

$\frac{du}{dx} = 2x$; $du = 2x\,dx$.

$= \int \frac{1}{2} u^7\,du$ ✓

$= \frac{1}{2}\int u^7\,du = \frac{1}{2}\left(\frac{u^8}{8}\right) + c$

$= \frac{u^8}{16} + c = \underline{\frac{(1+x^2)^8}{16} + c}$ ✓

c) Step 1

Let $n=1$.

$3^{2n}-1 = 3^2-1 = 8 = 8(1)$ which is divisible by 8

$\therefore$ true for $n=1$

Step 2

Assume true for $n=k$

$3^{2k}-1 = 8M$ (M is an integer)

R.T.P. also true for $n=k+1$

$3^{2k+2}-1 = 8N$ (N is another integer)

LHS $3^{2k+2}-1 = 3^2(3^{2k}-1)+8$

$= 9(8M)+8 = 8(9M+1)$

$= 8N = \text{RHS}$

Step 3

If $n=k$ is true and $n=k+1$ is true, ~~and~~ as well as $n=1$ is also true, then $n=1+1=2$ is true, $n=2+1=3$ is true and so on. $\therefore$ by TPOMI, it is true for all positive integers $n \geq 1$.

QUESTION 4

A) i.) For $Q(x)$ to be divisible by $2x+3$,

$Q(-1.5)=0$.

$Q(-1.5) = 4\left(-\frac{3}{2}\right)^3 + \frac{3}{2} + k = 0$

$4\left(-\frac{27}{8}\right) + \frac{3}{2} + k = 0$

$k = -\frac{3}{2} + \frac{27}{2}$; $k = \frac{24}{2} = 12$

$\therefore k = 12$

ii.) Long division of $4x^3+0x^2-x+12$ by $2x+3$:

$$
\begin{array}{r l}
 & 2x^2-3x+4 \\
2x+3 & \overline{)\,4x^3+0x^2-x+12} \\
 & 4x^3+6x^2 \\
\hline
 & -6x^2-x \\
 & -6x^2-9x \\
\hline
 & 8x+12 \\
 & 8x+12 \\
\hline
 & 00
\end{array}
$$

$\therefore \underline{Q(x) = (2x+3)(2x^2-3x+4)}$

B) $x^2 = 4y$

$y = \frac{x^2}{4}$; $\frac{dy}{dx} = \frac{x}{2}$.

Grad. of tgt at $P = \frac{2p}{2} = p$

$\therefore$ grad. of normal at $p = -\frac{1}{p}$ ($m_1 m_2 = -1$ for $\perp$ lines)

Eqt of normal at P is:

$(y-p^2) = -\frac{1}{p}(x-2p)$

$p(y-p^2) + x - 2p = 0$

$py - p^3 + x - 2p = 0$; $\underline{x = -py + 2p + p^3}$

c) $6\cos^2\theta + 8\cos\theta\sin\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\sin 2\theta = 2\sin\theta\cos\theta$

i.) $6\cos^2\theta + 8\cos\theta\sin\theta$

$= 3(2\cos^2\theta - 1) + 3 + 4\sin 2\theta$

$= \underline{3\cos 2\theta + 4\sin 2\theta + 3}$

ii.) $[3\cos 2\theta + 4\sin 2\theta] + 3 = R\cos(2\theta - \alpha) + 3$.

$= R(\cos 2\theta\cos\alpha + \sin 2\theta\sin\alpha) + 3$

$3 = R\cos\alpha$ — (1)

$4 = R\sin\alpha$ — (2)

$\frac{(2)}{(1)} = \frac{R\sin\alpha}{R\cos\alpha} = \tan\alpha = \frac{4}{3}$

$\underline{\alpha = 53°8'}$

$R = \sqrt{a^2+b^2} = \sqrt{9+16} = 5$

$\therefore \quad 3\cos 2\theta + 4\sin 2\theta + \cancel{3} = 5\cos(2\theta - 53°8') + \cancel{3}$ ✓ 3

$\equiv 5\cos(2\theta - 53°8') + A$

$\therefore \quad 3\cos 2\theta + 4\sin 2\theta = 5\cos(2\theta - 53°8')$

iii.) $6\cos 2\theta + 8\sin\theta\cos\theta = 5\cos(2\theta - 53°8')^{+3} = 4 \qquad 0° \leq \theta \leq 360°$

$\cos(2\theta - 53°8') = \frac{4}{5}$ ✓ $\quad 0 - 53°8' \leq 2\theta - 53°8' \leq 720° - 53$

$2\theta - 53°8' = \cancel{-36°52'}, \cancel{36°52'}, \cancel{323°8'}, 396°52';$

~~168°8'~~ $\quad 75°28', \; 281°32', \; 435°27, \; 641°32'$

$2\theta = 16°16', \; 90°, \; 376°16', \; 450°$

$\boxed{\theta = 8°8', \; 45°, \; 188°8', \; 225°}$

$491°35'$

$2\theta = 131°36', \; 334°40', \; \& \; 694°40$

$\theta = 65°48', \; 167°20', \; 245°47',$

$347°20'.$

QUESTION 5

A.) i.) $P = P_0 e^{kt}$

$\frac{dP}{dt} = kP_0 e^{kt} = kP.$ ✓

ii.) when $t = 0$, $P = P_0$ $\quad \therefore P_0$ is initial population.

when $t = 3$, $P = 2P_0$.

$2P_0 = P_0 e^{3k} \; ; \; e^{3k} = 2$

$3k = \ln 2$

$k = \frac{\ln 2}{3}$ ✓

$\therefore \quad P = P_0 e^{\frac{\ln 2}{3}t}$

$10P_0 = P_0 e^{\frac{\ln 2}{3}t}$ ✓

$e^{\frac{\ln 2}{3}t} = 10 \; ; \quad \frac{\ln 2}{3}t = \ln 10$ ✓

$t = \frac{\ln 10 \times 3}{\ln 2}$ ✓ $= 9\text{hrs } 58\text{min}$

$\therefore$ $\boxed{\text{at } 10.58\text{am}}$

B) $v = (2x+1)$ cm/s

$$\frac{dx}{dt} = 2x+1$$

$$\frac{dt}{dx} = \frac{1}{2x+1}$$

$$t = \frac{1}{2}\ln(2x+1) + c$$

when $t=0$, $x=1$

$0 = \frac{1}{2}\ln 3 + c$; $c = -\frac{1}{2}\ln 3 = -\ln\sqrt{3}$.

$$t = \frac{1}{2}\ln(2x+1) - \ln\sqrt{3}$$ ✓

$$t + \ln\sqrt{3} = \frac{1}{2}\ln(2x+1)$$

$$2t + 2\ln\sqrt{3} = \ln(2x+1)$$

$$2x+1 = e^{2t+2\ln\sqrt{3}}$$

$$x = \frac{e^{2t+2\ln\sqrt{3}} - 1}{2}$$ /

$$x = \frac{e^{2t} \cdot e^{\ln 3} - 1}{2}$$

$$\boxed{x = \frac{3e^{2t} - 1}{2}}$$ /

2

c) PX = PY (Tangents drawn from ~~a~~ pt outside a circle ~~drawn~~ are equal)

∴ ΔPXY is an isos Δ (2 sides equal) /

∴ $\angle PXY = \angle PYX$ (base $\angle$ of isos Δ are equal)

Let $\angle PXY = x$

$\angle PXY = \angle XZY = x$ (angle in alt. segment theorem).

$\angle YXZ = \angle XPY = 2a$ (given) ✓

In Δ PXY, $2a + x + x = 180°$ ($\angle$ sum $\Delta = 180°$)

In ΔXZY, $2a + x + \angle XYZ = 180°$ (" ") ✓

$\angle XYZ = x$

$\therefore \angle XYZ = \angle XZY = x$

$\therefore XZ = XY$ (ΔXZY is an isos Δ. base $\angle$ equal. 2 sides also equal)

QUESTION 6

A) $V = \pi \int_0^{\pi} \sin^2 x \, dx$

$= \frac{\pi}{2} \int_0^{\pi} 1 - \cos 2x \, dx$

$= \frac{\pi}{2} \left(x - \frac{1}{2} \sin 2x \right)_0^{\pi}$

$= \frac{\pi}{2} \left(\pi - \frac{\sin 2\pi}{2} \right)$

$\boxed{= \left(\frac{\pi^2}{2} \right) u^3}$

B) i) $\ddot{x} = 0$

$\dot{x} = V\cos\alpha$

$x = Vt\cos\alpha$.

$\ddot{y} = -g$

$\dot{y} = -gt + V\sin\alpha$

$y = -\frac{gt^2}{2} + Vt\sin\alpha$.

ii) $x = Vt\cos\alpha$

$x = 400t\cos 0$

$x = 400t$

$y = -\frac{10t^2}{2} + 400t\sin 0 + 1280$

$y = -5t^2 + 1280$

d) Find t when $y = 0$

$1280 - 5t^2 = 0$; $5t^2 = 1280$

$t^2 = 256$

$t = 16s \; (t > 0)$

$x = 400(16) = 6400 \text{ m} = 6.4 \text{ km}$ before Q.

$6400 = V \cdot \frac{1}{2}$

β) Find trajectory of package.

$x = 400t$; $t = \frac{x}{400}$

$$y = -5\left(\frac{x^2}{160{,}000}\right) + 1280$$

$$y = 1280 - \frac{x^2}{32{,}000}.$$

$$\frac{dy}{dx} = \frac{-2x}{32{,}000} = \frac{-x}{16{,}000}$$

$V = \frac{1280}{6400}$

$\frac{dx}{dt}$

$\frac{dx}{dt} = 400$

$\dot{x} = V$, $\dot{y} = -gt$

$\frac{\quad 1280}{6400}$

At $t = 16$

$\dot{x} = 400$, $\dot{y} = -16 \times 10$

$= -160$

Vel. on impact $= \sqrt{\dot{x}^2 + \dot{y}^2}$

$= \sqrt{(400)^2 + (-160)^2}$

$= 430.81$

$= 431\ ms^{-1}$ as req^d.

QUESTION 7

A.)

i.) $\dfrac{5}{(x-2)(x+3)} = \dfrac{A}{x-2} + \dfrac{B}{x+3}$

$= \dfrac{A(x+3)+B(x-2)}{(x-2)(x+3)}$

$A(x+3)+B(x-2) = 5$ ✓

$(A+B)x + 3A-2B = 5$

$A+B=0; \quad A=-B \quad -①$

$3A-2B=5$. Sub in ①

$3(-B)-2B=5; \; -5B=5; \; \underline{B=-1}$ ✓

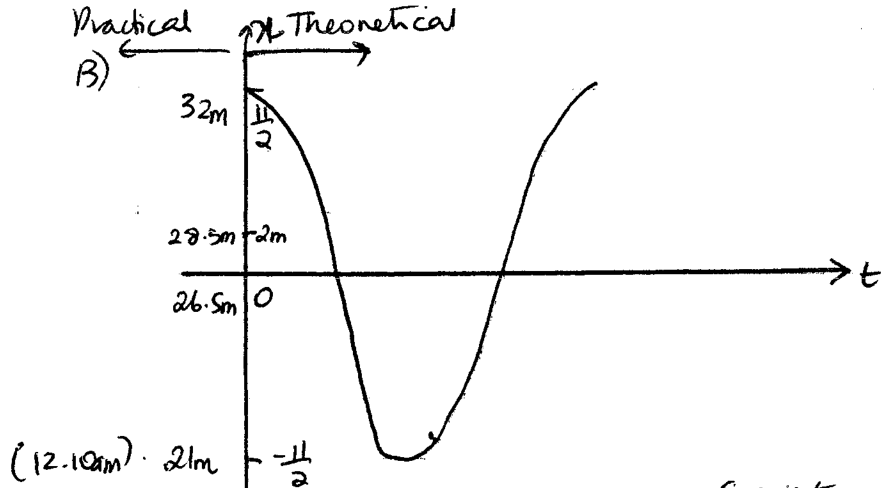
$\underline{A=1}$

$\therefore \dfrac{5}{(x-2)(x+3)} = \dfrac{1}{x-2} - \dfrac{1}{x+3}$. ✓

ii.) $\displaystyle\int \frac{5}{(x-2)(x+3)}\,dx = \int \frac{1}{x-2} - \frac{1}{x+3}\,dx$ ✓

$= \underline{\ln(x-2) - \ln(x+3) + C}$ ✓

B)



Period $= \dfrac{2\pi}{n} = 13$

$n = \underline{\underline{\dfrac{2\pi}{13}}}$ ✓

$x = a\cos nt$

$x = \dfrac{11}{2}\cos\dfrac{2\pi}{13}t$ ✓

$2 = \dfrac{11}{2}\cos\dfrac{2\pi}{13}t \quad ; \quad \cos\dfrac{2\pi}{13}t = \dfrac{4}{11}$ ✓

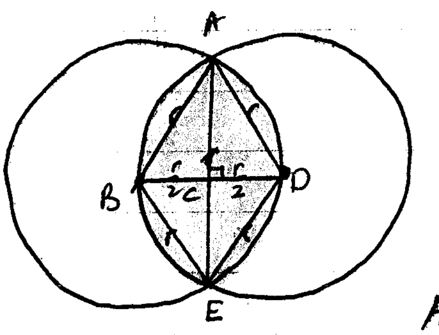
$\frac{2\pi}{13} t = 1.198$

$t = 2 \text{ hrs } 29 \text{ min}$

$\therefore$ earliest time at which ship can enter the harbour $=$

$12.10 \text{ am} + 2 \text{ hrs } 29 \text{ min} = \underline{\underline{2.39 \text{ am}}}$

c)



In $\triangle ACD$, $AC^2 = r^2 - \frac{r^2}{4}$ (pythag. theorem)

$AC^2 = \frac{3r^2}{4}$

$AC = \frac{\sqrt{3}r}{2}$ $(AC > 0)$

Area of $\triangle ACD = \frac{1}{2} \times \frac{r}{2} \times \frac{\sqrt{3}r}{2}$

$= \underline{\frac{\sqrt{3}r^2}{8}} u^2$

$\tan \angle AOC = \frac{\sqrt{3}\cancel{r}}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{r}}$

$\tan \angle AOC = \sqrt{3}$; $\underline{\angle AOC = \frac{\pi}{3}}$

$\angle AOC = \frac{\pi}{3}$

$\triangle ABC \equiv \triangle ACD \equiv \triangle CDE \equiv \triangle BCE$.

Area of Minor segment $= \dfrac{\left(\frac{\frac{2\pi}{3}}{2\pi} \times \pi r^2\right) - 4\left(\frac{\sqrt{3}r^2}{8}\right)}{2}$

$= \underline{\frac{\pi r^2}{6} - \frac{\sqrt{3}r^2}{4}}$

Area of shaded part $= 4\left(\frac{\sqrt{3}}{8}r^2\right) + 4\left(\frac{\pi r^2}{6} - \frac{\sqrt{3}r^2}{4}\right)$

$= \frac{\sqrt{3}r^2}{2} + \frac{2\pi r^2}{3} - \sqrt{3}r^2$

$= \frac{\sqrt{3}r^2 - 2\sqrt{3}r^2}{2} + \frac{2\pi r^2}{3}$

$\boxed{= r^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) u^2}$