Sydney Girls' High School

November 2009

MATHEMATICS EXTENSION 1

YEAR 11
ASSESSMENT TASK 1 for the 2010 HSC

Time Allowed: 75 minutes

TOPICS: Locus and Integration

Directions to Candidates

- There are five (5) questions.
- Attempt ALL questions.
- Questions are of equal value.
- Start each question on a new page.
- Write on one side of the paper only.
- Show all necessary working. Marks will be deducted for careless or badly arranged work.
- Diagrams are NOT drawn to scale.
- Board-approved calculators may be used.

Total: 75 marks

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

QUESTION 1 (15 marks) — Marks

a) Write down the indefinite integral of each of the following

(*i*) x^5 — 1

(*ii*) $\frac{5}{x^2}$ — 2

b) Find

(*i*) $\int (x^2 - 2)\ dx$ — 1

(*ii*) $\int (2-5x)^4\ dx$ — 2

(*iii*) $\int \frac{2x^2 + x}{x}\ dx$ — 2

c) A parabola has equation $x^2 = 8y$.

(*i*) Find the co-ordinates of the focus — 1

(*ii*) Find the equation of the directrix — 1

d) $\frac{d^2y}{dx^2} = -6x$ and when $x = 1$, $\frac{dy}{dx} = 1$ and $y = 1$. Find y when $x = 0$. — 5

QUESTION 2 (15 marks) — Marks

a) The focus of a parabola is $S\ (2,3)$ and its directrix is the line $y = -1$

(*i*) Sketch the parabola and indicate the co-ordinates of the vertex V. — 2

(*ii*) Write down the focal length of the parabola. — 1

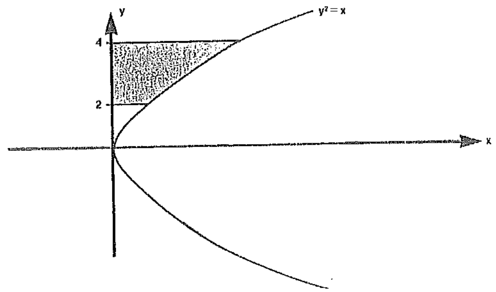
(*iii*) Find the equation of the parabola — 1

b) Evaluate:

(*i*) $\int_{-2}^{-1} \left(x - \frac{1}{x}\right)^2 dx$ — 3

(*ii*) $\int_0^4 \sqrt{t}\,(4-t)\ dt$ — 3

c) Find the area bounded by the curve $y^2 = x$, the y axis and the line $y = 2$ and $y = 4$. — 2



d) The table shows the values of a function $f(x)$ for five values of x. — 3

x	1	1.5	2	2.5	3
$f(x)$	5	1	-2	3	7

Use Simpson's rule with five function values to estimate $\int_1^3 f(x)\ dx$.

QUESTION 3 (15 marks)

Marks

a) If $\int_0^k (4-2x)\,dx = 4$ find the value of k. 4

b) Let A and B be the fixed points (2,1) and (-4,-5) and let P be the variable point (x, y)

(*i*) Write down expressions for PA^2 and PB^2 in terms of x and y. 2

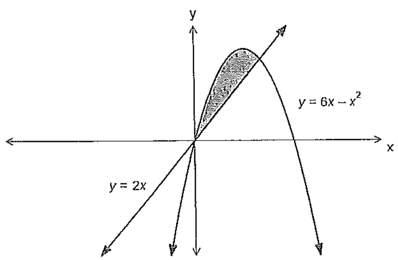
(*ii*) Suppose that P moves so that PA = 2PB. Deduce that P moves on a circle. 4

(*iii*) Find the centre and radius of this circle. 2

c) Find the area enclosed by $y = x^2 + x$ and the x-axis 3

QUESTION 4 (15 marks)

a) Given



(*i*) Find the points of intersection of the line $y = 2x$ and $y = 6x - x^2$ 2

(*ii*) Find the shaded area bounded by $y = 2x$ and $y = 6x - x^2$ 3

b) Given the line $y = x$ find the locus of the set of points exactly 2 units from this line. 4

c) A parabola has equation $y^2 - 4y - 4x - 8 = 0$.

(*i*) Express this equation in the form $(y-k)^2 = 4a(x-h)$ 2

(*ii*) Draw a sketch of the parabola indicating each of the following

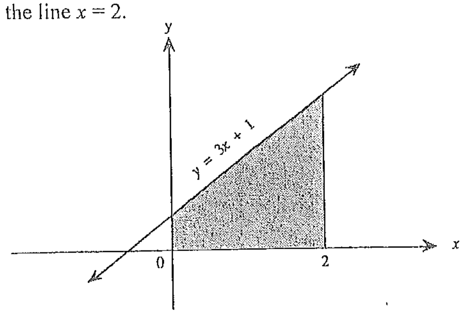
- Co-ordinates of the focus 1
- Co-ordinates of the vertex 1
- Equation of the directrix 1
- Equation of the axis of symmetry 1

QUESTION 5 (15 marks)

Marks

a) Use the trapezoidal rule with four sub-intervals (5 function values) to estimate $\int_0^1 \sqrt{1-x^2}\,dx$ to four decimal places. 3

b) The shaded region in the diagram is bounded by the line $y = 3x + 1$, the x-axis, the y axis and the line $x = 2$.



Calculate the volume of the solid of revolution formed when this region is rotated about the x axis. 3

c) Given A(-a, 0) and B(a,0) find the locus of P(x,y) if AP is perpendicular to BP. 3

d) Given the curves $y = x^2$ and $y = 8 - x^2$

(*i*) Find the points of intersection of the curves 2

(*ii*) On the same set of axes, draw a neat sketch of the curves $y = x^2$ and $y = 8 - x^2$ showing their points of intersection. 1

(*iii*) Hence, find the volume of the solid of revolution formed when the region between the curves $y = x^2$ and $y = 8 - x^2$ in the $x - y$ plane is rotated about the Y-axis. 3

THE END

Year 11

Assessment Task (1) - Extension 1

November 2009 for 2010 HSC.

Question 1

a) i) $\frac{x^6}{6} + C$ (1)

ii) $\frac{5x^{-1}}{-1} + C$

$= \frac{-5}{x} + C$ (2)

b) i) $\int x^2 - 2\,dx = \frac{x^3}{3} - 2x + C$ (1)

ii) $\int (2-5x)^4\,dx = \frac{(2-5x)^5}{-5.5} + C$

$= \frac{(2-5x)^5}{-25} + C$ (2)

iii) $\int \frac{2x^2+x}{x}\,dx$

$= \int 2x+1\,dx$

$= \frac{2x^2}{2} + x + C$

$= x^2 + x + C$ (2)

c) $x^2 = 8y$

i) $4a = 8$

$a = 2$ (1)

$\therefore$ Focus $(0,2)$

ii) $y = -2$ (1)

(11)

d) $\frac{d^2y}{dx^2} = -6x$

$\frac{dy}{dx} = -\frac{6x^2}{2} + C$

$\frac{dy}{dx} = -3x^2 + C$

at $x = 1$, $\frac{dy}{dx} = 1$

$-3(1)^2 + C = 1$

$-3 + C = 1$

$C = 4$ (5)

$\frac{dy}{dx} = -3x^2 + 4$

$y = \frac{-3x^3}{3} + 4x + C$

$y = -x^3 + 4x + C$

at $x = 1$, $y = 1$

$1 = -(1)^3 + 4(1) + C$

$1 = -1 + 4 + C$

$1 = 3 + C$

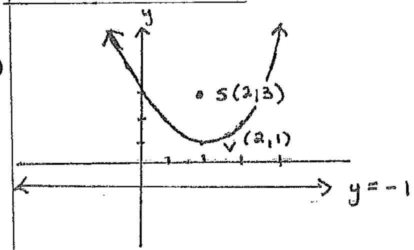
$\therefore 1 - 3 = C \quad C = -2$

$y = -x^3 + 4x - 2$

at $x = 0$ $\quad y = -2$

Question 2

a)



V (2,1) (2)

b) focal length = 2 units (1)

c) $(x-h)^2 = 4a(y-k)$

$(x-2)^2 = 8(y-1)$ (1)

b) i) $\int_{-2}^{-1} (x - \frac{1}{x})^2\,dx$

$= \int_{-2}^{-1} x^2 - 2 + \frac{1}{x^2}\,dx$

$= \left[\frac{x^3}{3} - 2x + \frac{x^{-1}}{-1}\right]_{-2}^{-1}$

$= \left[\frac{x^3}{3} - 2x - \frac{1}{x}\right]_{-2}^{-1}$

$= \left[\left(\frac{-1}{3} + 2 + 1\right) - \left(\frac{-8}{3} + 4 + \frac{1}{2}\right)\right]$

$= \left[2\,^2/_3 - 1\,^5/_6\right]$

$= {}^5/_6$ (3)

ii) $\int_0^4 \sqrt{t}\,(4-t)\,dt$

$= \int_0^4 4\sqrt{t} - t^{3/2}\,dt$

$= \int_0^4 4t^{1/2} - t^{3/2}\,dt$

ii) $= \left[\frac{4t^{3/2}}{3/2} - \frac{t^{5/2}}{5/2}\right]_0^4$

$= \left[\frac{2}{3} 4t^{3/2} - \frac{2}{5} t^{5/2}\right]_0^4$

$= \left[\frac{2}{3} 4(4)^{3/2} - \frac{2}{5}(4)^{5/2}\right] - 0$

$= 21\,^1/_3 - 12\,^4/_5$

$= 8\,^8/_{15}.$ (3)

c) $\int_2^4 x\,dy = \int_2^4 y^2\,dy$

$= \left.\frac{y^3}{3}\right]_2^4$

$= \frac{4^3}{3} - \frac{2^3}{3}$

$= 21\,^1/_3 - {}^8/_3$

$= 18\,^2/_3$ units^2 (2)

d)

x	$f(x)$	w	$w.f(x)$
1	5	1	5
1.5	1	4	4
2	-2	2	-4
2.5	3	4	12
3	7	1	7
		Total	24

$\therefore A \doteqdot \frac{h}{3} \times \Sigma f(x).w$

$\doteqdot \frac{1/2}{3} \times 24$

$\doteqdot 4$ (3)

Question 3

a) $\int_0^k (4-2x)\,dx = 4$

$\left[4x - \frac{2x^2}{2}\right]_0^k = 4$

$\left[4x - x^2\right]_0^k = 4$

$4k - k^2 = 4$

$k^2 - 4k + 4 = 0$

$(k-2)(k-2) = 0$

$k = 2$ (4)

b) $A(2,1)$ $B(-4,-5)$

$P(x,y)$

i) $PA = \sqrt{(x-2)^2 + (y-1)^2}$

$PA^2 = (x-2)^2 + (y-1)^2$

$PB = \sqrt{(x+4)^2 + (y+5)^2}$

$PB^2 = (x+4)^2 + (y+5)^2$ (2)

ii) $PA = 2PB$

$PA^2 = 4PB^2$

$(x-2)^2 + (y-1)^2 = 4\left[(x+4)^2 + (y+5)^2\right]$

$x^2 - 4x + 4 + y^2 - 2y + 1 =$
$4\left[x^2 + 8x + 16 + y^2 + 10y + 25\right]$

$x^2 - 4x + y^2 - 2y + 5 =$
$4\left[x^2 + 8x + y^2 + 10y + 41\right]$

$3x^2 + 36x + 3y^2 + 42y + 159 = 0$

$x^2 + 12x + y^2 + 14y + 53 = 0$

hence
$x^2 + 12x + y^2 + 14y + 53 = 0$
is a circle. (4)

iii) $x^2 + 12x + 6^2 + y^2 + 14y + 7^2 = -53 + 85$

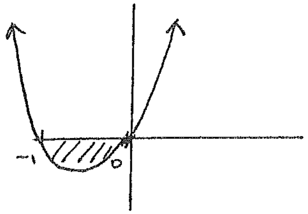
$(x+6)^2 + (y+7)^2 = 32$

Centre $(-6,-7)$

radius $= \sqrt{32}$

$= 4\sqrt{2}$ units (2)

c) $y = x^2 + x \rightarrow y = x(x+1)$



$\text{Area} = \left| \int_{-1}^{0} x^2 + x \, dx \right|$

$= \left| \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{0} \right|$

$= \left| 0 - \left[-\frac{1}{3} + \frac{1}{2}\right] \right|$

$= \left| -\frac{1}{6} \right|$

$= \frac{1}{6}$ units ^{2} (3)

Question 4

a) i) $y = 2x$

$y = 6x - x^2$

$6x - x^2 = 2x$

$4x - x^2 = 0$

$x(4-x) = 0$

$x = 0$ or $x = 4$

$y = 0$ or $y = 8$

$\therefore$ Points of intersection

$P_1(0,0)$ & $P_2(4,8)$ (2)

ii) $\int_0^4 6x - x^2\,dx - \int_0^4 2x\,dx$

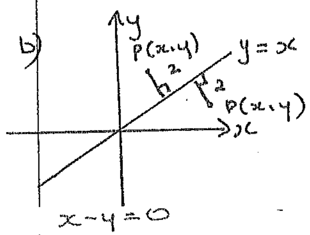
$= \int_0^4 4x - x^2\,dx$

$= \left[2x^2 - \frac{x^3}{3}\right]_0^4$

$= \left(32 - \frac{64}{3}\right) - 0$ (3)

$= 10\tfrac{2}{3}$ units ^{2}

b)



Perpendicular dist. = 2

$2 = \frac{|x-y|}{\sqrt{1^2+(-1)^2}}$

$2 = \frac{|x-y|}{\sqrt{2}}$

$2 = \frac{x-y}{\sqrt{2}}$ if $x > y$

$\therefore 2\sqrt{2} = x - y$

$x - y - 2\sqrt{2} = 0$ (4)

OR

$2\sqrt{2} = y - x$ if $x < y$

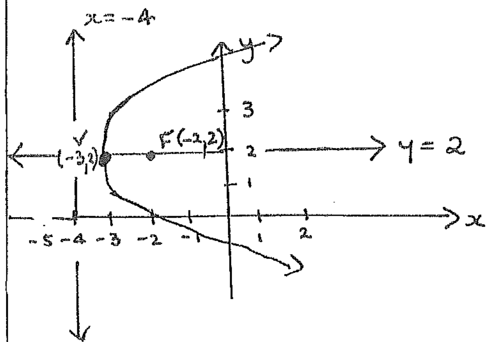
$x - y + 2\sqrt{2} = 0$

c) $y^2 - 4y - 4x - 8 = 0$

i) $y^2 - 4y + (-2)^2 = 4x + 8 + 4$

$(y-2)^2 = 4(x+3)$ (2)

ii)
- Focus $a = 1$
  $(-2, 2)$ (1)
- Vertex (h,k) $a = 1$
  $(-3, 2)$ (1)
- $x = -4$ (1)
- Axis of symmetry
  $y = 2$ (1)



Question 5

a)

x	fx	w	$w\,fx$
0	1	1	1
$\frac{1}{4}$	$\frac{\sqrt{15}}{4}$	2	$\frac{\sqrt{15}}{2}$
$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	2	$\sqrt{3}$
$\frac{3}{4}$	$\frac{\sqrt{7}}{4}$	2	$\frac{\sqrt{7}}{2}$
1	0	1	0

$$\Sigma(w.fx) = 1 + \frac{\sqrt{15}}{2} + \sqrt{3} + \frac{\sqrt{7}}{2}$$
$$\doteqdot 5.99$$

$$\therefore \int_0^1 \sqrt{1-x^2}\,dx \doteqdot \frac{\frac{1}{4}}{2}(5.9914..)$$

$$\doteqdot 0.7489$$

(to 4 decimal places)

3

b)

$$V = \pi\int_0^2 y^2\,dx$$
$$= \pi\int_0^2 (3x+1)^2\,dx$$
$$= \pi\int_0^2 9x^2 + 6x + 1\,dx$$
$$= \pi\left[\frac{9x^3}{3} + \frac{6x^2}{2} + x\right]_0^2$$
$$= \pi\left[3x^3 + 3x^2 + x\right]_0^2$$
$$= \pi\left[24 + 12 + 2 - 0\right]$$
$$= 38\pi \text{ units}^3$$

3

c) $A(-a,0)$ and $B(a,0)$

$P(x,y)$

Gradient $AP = \dfrac{y}{x+a}$

Gradient $BP = \dfrac{y}{x-a}$

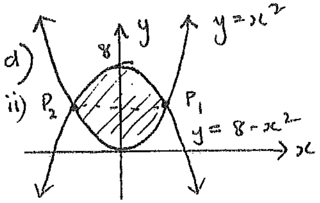
if $AP \perp BP \therefore m_1 \times m_2 = -1$

$$\frac{y}{x+a} \times \frac{y}{x-a} = -1$$
$$y^2 = -1(x+a)(x-a)$$
$$y^2 = -1(x^2 - a^2)$$
$$y^2 = -x^2 + a^2$$
$$\therefore x^2 + y^2 = a^2$$

3

d)

(ii)



1

i) Points of intersection

$x^2 = 8 - x^2$ $\quad y = (2)^2 = 4$

$2x^2 = 8$ $\quad y = (-2)^2 =$

$x^2 = 4$

$x = \pm 2$

2

$\therefore P_1(2,4)$ and $P_2(-2,4)$

iii)
$$V = 2\pi\int_0^4 x^2\,dy$$
$$= 2\pi\int_0^4 y\,dy$$
$$= 2\pi\left[\frac{y^2}{2}\right]_0^4$$
$$V = 2\pi \times 8 = 16\pi \text{ units}^3$$

3