Sydney Girls High School

Year 11
MATHEMATICS EXTENSION 1

2012 YEARLY EXAMINATION

Time Allowed: 80 minutes + 5 minutes Reading Time

TOPICS: Harder 2U, Differential Calculus, Sequences and Series, Probability, The Quadratic Polynomial and Induction.

Directions to Candidates

- There are four (4) questions.
- Attempt ALL questions.
- Questions are of equal value.
- Start each question on a new page.
- Write on one side of the paper only.
- Show all necessary working. Marks will be deducted for careless or badly arranged work.
- Diagrams are NOT drawn to scale.
- Board-approved calculators may be used.

Name: ______________________ **Teacher:** _

Question 1 **18 Marks**

a) The quadratic equation $3x^2-2x-5=0$ has roots α and β. Evaluate:

i. $\alpha+\beta$ 1

ii. $\alpha\beta$ 1

iii. $\alpha\beta^2+\beta\alpha^2$ 2

iv. $(\alpha-\beta)^2$ 2

b) Find the coordinates of the point $P(x,y)$ that divides the interval AB joining points $A(-5,11)$ and $B(7,3)$, externally in the ratio 3:1. 2

c) Given $4x^2-5x+6\equiv a(x-2)^2+b(x+3)+c$ find the values of a, b and c. 3

d) Solve $\frac{4}{5-x}\geq 1$. 3

e) Differentiate the following:

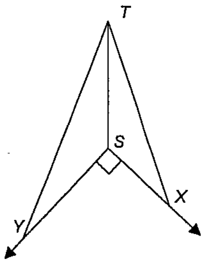
i. $\frac{1+x^3}{x^2}$ 2

ii. $\left(5-x^3\right)\sqrt{5-x^3}$ 2

Question 2 (Begin on a New Page) **18 Marks**

a) Solve the equation $7(7)^{2x}-8(7)^{x}+1=0$. **3**

b) A flagpole, ST, is h metres high and on level ground. From a point, X, due south of the pole, the angle of elevation to T, the top of the pole, is 45°. From a point, Y, due west of the pole, the angle of elevation of the top of the pole is 30°.



i. Copy the given diagram, showing all of the above information. **1**

ii. Find the exact value of the height of the pole, if the distance between X and Y is 10 metres. **3**

c) In a certain course all students sit for a theory examination in which 70% of the candidates pass. Those who pass the theory examination then sit a practical test, which is passed by 40% of the candidates.

A student is chosen at random. Find the probability that:

i. the student passes both examinations **2**

ii. the student passes just one of the examinations. **2**

d) For the function $f(x)=\dfrac{-2}{x^2+1}$:

i) Evaluate $f(0)$. **1**

ii) Show that $f(x)$ is an even function. **1**

iii) What happens to the values of $f(x)$ as x gets very large? **1**

iv) Find the domain and range of $f(x)$. **2**

v) Draw a sketch of the function. **2**

Question 3 (Begin on a New Page) **18 Marks**

a) Consider the geometric sequence $\sin^2 x, \sin^4 x, \sin^6 x...$ where $0<x<90°$. **3**

Find the limiting sum of the sequence, expressing the answer in simplest form.

b) Consider the circle $x^2+y^2-2x-14y+25=0$.

i. Find the centre and radius of the circle. **2**

ii. Show that, if the line $y=mx$ intersects the circle in two distinct points, then $(1+7m)^2-25(1+m^2)>0$. **3**

iii. For what values of m is the line $y=mx$ a tangent to the circle? **2**

c) Christina borrows \$480,000 from a finance company to buy a house.

She pays interest at 6% per annum, calculated quarterly on the balance still owing.

The loan is to be repaid at the end of 20 years with equal quarterly repayments of \$$P$.

Let A_n = the amount owing after the nth repayment.

i. Show that after the first quarterly repayment of \$$P$ Christina owes an amount equivalent to $A_1=\$487,200-\$P$. **1**

ii. Find an expression for the amount still owing after 3 repayments of \$$P$. **2**

iii. Find the value of \$$P$ to the nearest cent. **2**

d) Use the Principle of Mathematical Induction to prove that, for all positive integers, $n\geq 1$, **3**

$$1\times 2^0+2\times 2^1+3\times 2^2+....+n\times 2^{n-1}=1+(n-1)2^n$$

Question 4 (Begin on a New Page) **18 Marks**

a) Show that $-3x^2+6x-7<0$ for all x. **2**

b) For the function, $y=f(x)$, you are given the following information: **3**

$$f'(2)=0;\ f(2)=-5;\ f(x)=f(-x);\ f(0)=10.$$

Draw a neat sketch of a possible curve, clearly showing all of the information given.

c)

i. Factorise $3x^3+3x^2-x-1$. **2**

ii. Hence solve $3\tan^3\theta+3\tan^2\theta-\tan\theta-1=0$ for $0\le\theta\le 360°$. **3**

d) Find $\lim_{h\to 0}\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}$ **3**

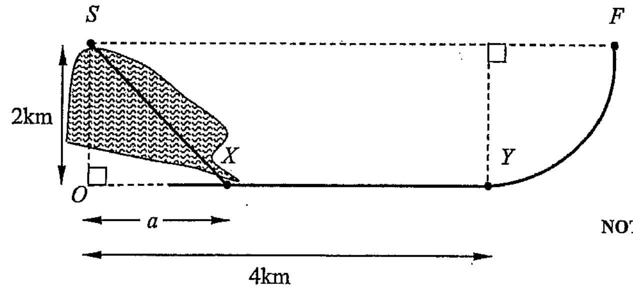
Question 4 continues on the next page

Question 4 (Continued) **Marks**

e) Helen is training to compete in a mini triathlon.

The course she practises on consists of three legs, which starts at S and finishes at F. The first leg is a straight swim from S to a point X. The second leg is a bike ride from X to Y along a straight road OY and the final leg is a jog from Y to F around a circular path. The perpendicular distance from S to O is 2km while the distance OY is 4km.

Helen can swim at 6km/h, bike ride at 12km/h and jog at 8km/h.



i. If the distance $OX=a$ km, show that the time T that it takes Helen to complete the three legs is given by $T=\dfrac{4\sqrt{a^2+4}-2a+(8+3\pi)}{24}$ hours. **2**

ii. Find the value of a, that will allow Helen to minimise the time taken to complete the three legs of her practise course. **3**

-- End Of Test --

YR 11 Extension One, 2012 YEARLY, SOLUTIONS

1 a) i. $-\frac{-2}{3} = \frac{2}{3}$ ✓

ii. $\frac{-5}{3}$ ✓

iii. $\alpha\beta(\beta+\alpha)$ ✓

$= -\frac{5}{3} \times \frac{2}{3}$

$= -\frac{10}{9}$ ✓

iv. $(\alpha+\beta)^2 - 4\alpha\beta$ ✓

$= \left(\frac{2}{3}\right)^2 - 4 \times -\frac{5}{3}$

$= \frac{64}{9}$ ✓

b) $\left(\frac{1\times-5-3\times7}{-3+1}, \frac{11\times1-3\times3}{-3+1}\right)$ ✓

$= \left(\frac{-26}{-2}, \frac{2}{-2}\right)$

$= (13, -1)$ ✓

c) $4x^2 - 5x + 6$

$\equiv a(x^2-4x+4) + bx + 3b + c$

$\equiv ax^2 - 4ax + 4a + bx + 3b + c$

$\equiv ax^2 + (b-4a)x + 4a + 3b + c$

$a = 4$ ✓ $\quad b - 4a = -5$ ✓ $\quad 4a + 3b + c = 6$ ✓

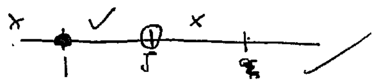
$b - 4\times4 = -5 \quad 16 + 33 + c = 6$

$b = 11 \quad c = -43$

d) $\frac{4}{5-x} = 1$

$4 = 5 - x$

$x = 1$ ✓



$1 \leq x < 5$ ✓

e) i. $y = x^{-2} + x$

$\frac{dy}{dx} = -2x^{-3} + 1$ ✓✓

ii. $y = (5 - x^3)^{\frac{3}{2}}$

$\frac{dy}{dx} = \frac{3}{2}(5-x^3)^{\frac{1}{2}} \times -3x^2$

$= -\frac{9x^2}{2}\sqrt{5-x^3}$ ✓✓

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Question Two:

a) $7(7)^{2x} - 8(7)^{x} + 1 = 0$

Let $m = 7^x$

$7m^2 - 8m + 1 = 0$

$7m^2 - 7m - m + 1 = 0$

$7m(m-1) - 1(m-1) = 0$

$(m-1)(7m-1) = 0$ ✓

$7m - 1 = 0$ or $m - 1 = 0$

$7m = 1$ $m = 1$

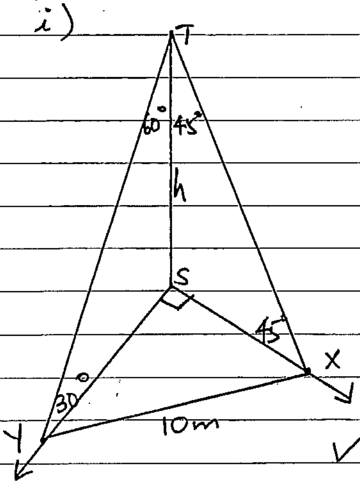
$m = \frac{1}{7}$ $7^x = 1$

$7^x = 7^{-1}$ $7^x = 7^0$

$\therefore x = -1$ or $x = 0$ ✓✓

b)

i)



✓

ii) In ΔSTY: $\tan 60° = \frac{SY}{h}$

$SY = h \tan 60°$

$\therefore SY = \sqrt{3}h$ ✓

In ΔSTX: $\tan 45° = \frac{SX}{h}$

$SX = h \tan 45°$

$\therefore SX = h$ ✓

In ΔSXY: $XY^2 = SY^2 + SX^2$ (Pythagoras)

$10^2 = (\sqrt{3}h)^2 + h^2$

$100 = 3h^2 + h^2$

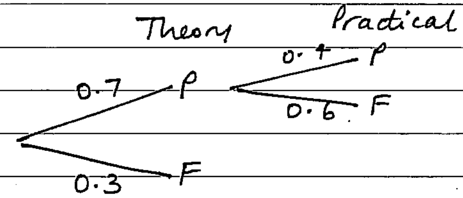
$100 = 4h^2$

$\therefore h^2 = 25$ ✓

$h = 5m$ $(h > 0)$

Question Two:

c)



i) $P(PP) = 0.7 \times 0.4$ ✓

$= 0.28$ or $\frac{7}{25}$ or 28%. ✓

ii) $P(PF)$

$= 0.7 \times 0.6$ ✓

$= 0.42$ or $\frac{21}{50}$ or 42%. ✓

d) i) $f(0) = -\frac{2}{1} = -2$ ✓

ii) $f(-x) = \frac{-2}{(-x)^2+1} = \frac{-2}{x^2+1} = f(x)$

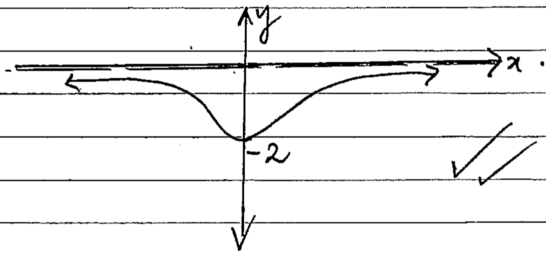
$\therefore f(-x) = f(x)$ $\therefore$ even. ✓

iii) as $x \to \infty$, $f(x) \to 0$ ✓

iv) D: all real x ✓

R: $-2 \le y < 0$ ✓

v)



✓✓

Question 3

a)

$$a = \sin^2 x$$
$$r = \frac{\sin^4 x}{\sin^2 x} = \sin^2 x$$

$$S_\infty = \frac{a}{1-r} = \frac{\sin^2 x}{1-\sin^2 x} = \frac{\sin^2 x}{\cos^2 x} = \tan^2 x$$

b) i.

$$x^2 - 2x + y^2 - 14y = -25$$
$$x^2 - 2x + 1 + y^2 - 14y + 49 = -25 + 1 + 49$$
$$(x-1)^2 + (y-7)^2 = 25$$

Centre (1, 7) and radius = 5 units.

ii.

$$x^2 + y^2 - 2x - 14y + 25 = 0 \quad ---(1)$$
$$y = mx --- (2)$$

Sub (2) into (1):

$$x^2 + (mx)^2 - 2x - 14(mx) + 25 = 0$$
$$x^2 + m^2x^2 - 2x - 14mx + 25 = 0$$
$$(1+m^2)x^2 - 2(1+7m)x + 25 = 0$$
$$\Delta = b^2 - 4ac$$
$$= [-2(1+7m)]^2 - 4(1+m^2)\cdot 25$$
$$= 4(1+7m)^2 - 100(1+m^2)$$

Two points of intersection (2 real roots) $\Delta > 0$

$$4(1+7m)^2 - 100(1+m^2) > 0$$
$$(1+7m)^2 - 25(1+m^2) > 0$$

iii. For line to be a tangent, $\Delta = 0$:

$$(1+7m)^2 - 25(1+m^2) = 0$$
$$1 + 14m + 49m^2 - 25 - 25m^2 = 0$$
$$24m^2 + 14m - 24 = 0$$
$$12m^2 + 7m - 12 = 0$$
$$(4m-3)(3m+4) = 0$$
$$m = \frac{3}{4} \text{ or } m = -\frac{4}{3}$$

c) i.

$$r = 0.06 \div 4 = 0.015$$

$$A_1 = 480000 \times 1.015 - P = 487200 - P$$

ii.

$$A_2 = A_1 \times 1.015 - P$$
$$= (480000 \times 1.015 - P)1.015 - P$$
$$= 480000 \times 1.015^2 - 1.015P - P$$
$$= 480000 \times 1.015^2 - P(1 + 1.015)$$
$$A_3 = A_2 \times 1.015 - P$$
$$= (480000 \times 1.015^2 - 1.015P - P)1.015 - P$$
$$= 480000 \times 1.015^3 - 1.015^2 P - 1.015P - P$$
$$= 480000 \times 1.015^2 - P(1 + 1.015 + 1.015^2)$$

iii.

$$A_n = 480000 \times 1.015^n - P(1 + 1.015 + 1.015^2 + \ldots + 1.015^{n-1})$$
$$A_{80} = 480000 \times 1.015^{80} - P(1 + 1.015 + 1.015^2 + \ldots + 1.015^{79})$$
$$\text{but } A_{80} = 0 \quad (\text{loan repaid})$$
$$0 = 480000 \times 1.015^{80} - P(1 + 1.015 + 1.015^2 + \ldots + 1.015^{79})$$
$$480000 \times 1.015^{80} = P(1 + 1.015 + 1.015^2 + \ldots + 1.015^{79})$$
$$= P \times \frac{1(1.015^{80} - 1)}{1.015 - 1}$$
$$P = \frac{0.015 \times 480000 \times 1.015^{80}}{1.015^{80} - 1}$$
$$= \$10343.20 \quad (\text{nearest cent})$$

d) Prove true for $n = 1$:

$$LHS = 1 \times 2^0 = 1$$

$$RHS = 1 + (1-1)2^0 = 1$$

$LHS = RHS$

$\therefore$ true for $n = 1$

Assume true for $n = k$

$$1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \ldots + k \times 2^{k-1} = 1 + (k-1)2^k$$

Prove true for $n = k+1$

$$RTP\ 1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \ldots + k \times 2^{k-1} + (k+1) \times 2^k = 1 + k2^{k+1}$$
$$LHS = 1 \times 2^0 + 2 \times 2^1 + 3 \times 2^2 + \ldots + k \times 2^{k-1} + (k+1) \times 2^k$$
$$= 1 + (k-1)2^k + (k+1) \times 2^k \quad \text{using assumption}$$
$$= 1 + k \cdot 2^k - 2^k + k \cdot 2^k + 2^k$$
$$= 1 + 2k \cdot 2^k$$
$$= 1 + k \cdot 2^{k+1}$$
$$= RHS$$

Proven true for $n = k+1$.

Proven true by mathematical induction

Question 4.

a) $-3x^2 + 6x - 7 < 0$

$a = -3$

< 0

$\Delta = 6^2 - 4(-3)(-7)$

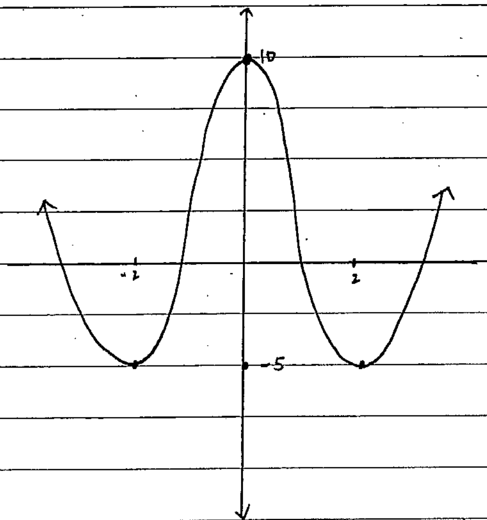
$= 36 - 84$

$= -48$

< 0

Since $a < 0$ and $\Delta < 0$, $-3x^2 + 6x - 7 < 0$ for all x.

b)



c) i) $3x^3 + 3x^2 - x - 1 = 3x^2(x+1) - 1(x+1)$

$= (3x^2 - 1)(x+1)$

ii) $(3\tan^2\theta - 1)(\tan\theta + 1) = 0$

$3\tan^2\theta = 1$

$\tan\theta = \pm 1/\sqrt{3}$

$\theta = 30°, 150°, 210°, 330°$

$\tan\theta = -1$

$\theta = 135°, 315°$

d. $\lim\limits_{h\to 0} \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim\limits_{h\to 0} \left\{ \dfrac{1}{x+h} - \dfrac{1}{x} \right\} \div \dfrac{h}{1}$

$= \lim\limits_{h\to 0} \dfrac{x - (x+h)}{x(x+h)} \times \dfrac{1}{h}$

$= \lim\limits_{h\to 0} \dfrac{-h}{hx(x+h)}$

$= \lim\limits_{h\to 0} \dfrac{-1}{x(x+h)}$

$= \dfrac{-1}{x^2}$

e. $SX^2 = 4 + a^2$ $\quad XY = 4 - a$ $\quad FY = 2 \times \pi \times 2 \times 1/4$

$SX = \sqrt{4 + a^2}$ $\quad\quad = \pi$

i) $T = \dfrac{\sqrt{4+a^2}}{6} + \dfrac{4-a}{12} + \dfrac{\pi}{8}$

$= \dfrac{4\sqrt{4+a^2} + 2(4-a) + 3\pi}{24}$

ii) $\dfrac{dT}{da} = \dfrac{\left[2(4+a^2)^{-1/2} \cdot 2a - 2\right] \cdot \cancel{24}}{24^{\cancel{2}}}$

for a min. time $\dfrac{dT}{da} = 0$

$0 = 4a(4+a^2)^{-1/2} - 2$

$2 = 4a(4+a^2)^{-1/2}$

$2(4+a^2)^{1/2} = 4a$

$4(4+a^2) = 16a^2$

$16 = 12a^2$

$a = \dfrac{2}{\sqrt{3}}$ $(a > 0)$

a	$\frac{2}{2\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	$\frac{4}{\sqrt{3}}$
$\frac{dT}{da}$	−	0	+
	\	—	/

$\therefore$ a min time occurs when

$a = \dfrac{2}{\sqrt{3}}$.