Sydney Girls High School

November 2012

MATHEMATICS
EXTENSION 1

YEAR 12
ASSESSMENT TASK 1 for HSC 2013

Time Allowed: 60 minutes
+ 5 minutes Reading Time

Topics: Locus, Integration, Mathematical Induction, Division of an interval in a given ratio, Other Inequalities

General Instructions:

- There are Seven (7) Questions which are not of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- Start each Question on a new page.
- Write on one side of paper only.
- Diagrams are NOT to scale.
- Board-approved calculators may be used.
- Write your student number clearly at the top of each question and clearly number each question.
- Note that: $\int x^n dx = \frac{1}{n+1}x^{n+1}; \quad n \neq -1$

Total: 50 marks

Student Name:________________ **Teacher Name:**____________

QUESTION 1 (7 Marks)

MARKS

a) Find a primitive function of each of the following:

i. $5x^3 - 4$ — 1

ii. $\frac{7-x^4}{x^2}$ — 2

iii. $(8x+3)^5$ — 2

b) Find the equation of the parabola, with vertical axis, vertex at the origin and passing through the point (6,3). — 2

QUESTION 2 (7 Marks)

MARKS

a) Given the equation of a parabola is $(x+4)^2 = -2(y-8)$, find:

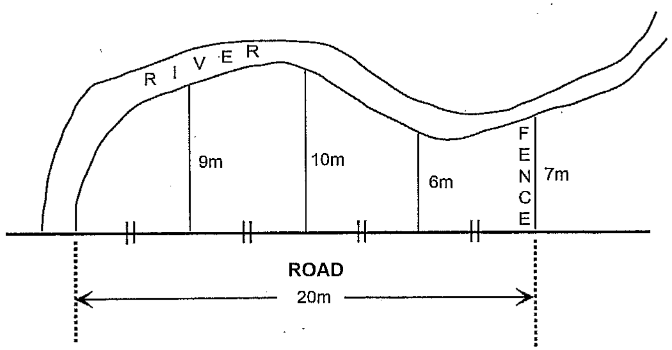
i. the vertex of the parabola. — 1

ii. the focal length. — 1

b) Find the equation of the parabola with focus (2,0) and directrix $x = -2$. — 2

c) A wasteland bordering a river bank and a straight road was fenced off and used as a recreational park. Perpendicular distances from the road to the river bank are shown on the diagram below.

Use Simpson's rule, with 5 function values, to approximate the area of the recreational park. — 3



QUESTION 3 (7 Marks)

MARKS

a) Solve the following: $\frac{x}{x-4} \le -3$. 3

b) i. Find the $x-$coordinates of the points of intersection of the line $2x+y=0$ and the parabola $y=x(x+4)$. 1

ii. Find the area bounded by $2x+y=0$ and $y=x(x+4)$. 3

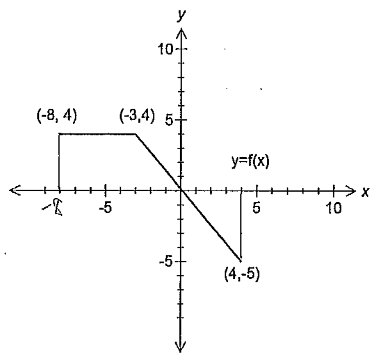
QUESTION 4 (7 Marks)

MARKS

a) Find in what ratio A divides BC, given that the points A(-1,3), B(0,1) and C(1,-1) lie in one line. 3

b) The point P(k,7) lies on the parabola $x^2=20y$.
Find the distance from P to the focus. 2

c) Given the graph of $y=f(x)$ below, find $\int_{-8}^{4} f(x)\,dx$. 2



QUESTION 5 (7 Marks)

MARKS

a) i. Draw a diagram showing the area bounded by $y=2x^3$, $y=2$ and the $y-$axis. 1

ii. The area is now rotated around the $x-$axis. Calculate the exact volume of the solid of revolution formed. 3

b) The vertex of a parabola is V(-1,4) and its directrix is $x=3$.

i. Find the coordinates of the focus. 1

ii. Find the equation of the parabola. 1

iii. Hence, sketch the parabola, showing the main features. 1

QUESTION 6 (7 Marks)

MARKS

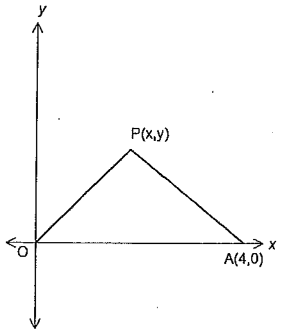
a) Prove by mathematical induction that $1^2+3^2+\ldots+(2n-1)^2=\frac{1}{3}n(2n-1)(2n+1)$ for $n\ge 1$ 4

b) Evaluate $\int_{-\frac{8}{3}}^{-1} \frac{2}{\sqrt{1-3x}}\,dx$ 3

QUESTION 7 (8 Marks)

MARKS

a)



i. A point $P(x,y)$ moves such that $\angle OPA$ is a right angle. Show that the locus of P is given by: $x^2+y^2-4x=0$. 2

ii. The locus of P is a circle, find its centre and radius. 1

b) A hollow vessel is formed when the parabola $9x^2=4y$ is rotated about the y-axis. The distance across the open end of the vessel is 1.5 metres.

i. Show that, when the depth of water in the vessel is h metres, the volume of water is $\dfrac{2\pi h^2}{9}$ m^3. 3

ii. Find the depth of the water in the vessel when it is half full. 2

End of Paper

Question 7:

a.

i. $m_{OP} = \dfrac{y}{x}$

$m_{AP} = \dfrac{y}{x-4}$

$m_{OP} \times m_{AP} = -1$

$\dfrac{y}{x} \times \dfrac{y}{x-4} = -1$

$y^2 = -x(x-4)$

$y^2 + x^2 - 4x = 0$

/2

ii. $y^2 + x^2 - 4x + 4 = 4$

$y^2 + (x-2)^2 = 4$

centre $(2,0)$ radius $= 2$

/1

b.

i. $V = \pi \int_0^h x^2 \, dy$

$= \pi \int_0^h \dfrac{4y}{9} \, dy$

$= \pi \left[\dfrac{2y^2}{9} \right]_0^h$

$= \dfrac{2\pi h^2}{9}$

/3

ii. When vessel is full:

$x = \dfrac{3}{4}$

$y = \dfrac{9}{4} \times \left(\dfrac{3}{4} \right)^2$

$= \dfrac{81}{64}$

$V = \dfrac{2\pi}{9} \times \left(\dfrac{81}{64} \right)^2$

$= \dfrac{729\pi}{2048}$

When vessel is half full:

$V = \dfrac{729\pi}{4096}$

$\dfrac{729\pi}{4096} = \dfrac{2\pi h^2}{9}$

$h^2 = \dfrac{6561}{8192}$

$h = 0.89$

/2

Extension One Task One: Feedback December 2012

Question	Comment
1	a) For indefinite integrals, don't forget to write "+C". b) Students who used a simple diagram and wrote the formula $x^2 = 4ay$ were more successful than those who did not.
2	a) Focal length is $\frac{1}{2}$ not $a = -\frac{1}{2}$ b) Students who used a diagram were more successful than those who did not. Athough $x = -2$ was confused with $y = -2$ P.L. c) Tabular Method (or variation) worked better than others
4.	b) k can be found by substitution c) better to use area formulae rather than calculus
3.	a) Note that $x = 4$ is NOT a solution to the original inequation and must be excluded. b)(ii) Most efficient solution uses $\int (f(x) - g(x))\,dx$ where $y = f(x)$ is the upper curve.
5	a) Alternate solution: $V =$ cylinder $-$ area under the curve. b) Sketching the parabola should include any intercepts; in this case $x = -2$.
6	a) Better solutions factorised the LHS expression in step 3. b) Most common mistake was the integration step (forgetting to divide by -3)
7	b ii) Many students incorrectly interpreted the vessel being half full as meaning the height was $\frac{h}{2}$ rather than the volume being $V/2$, which lead to no marks being awarded.

Nov. 2012 Ext. 1 Assessment Task 1 SOLUTIONS

Question 1

a) i) $\int (5x^3 - 4)\,dx = \frac{5x^4}{4} - 4x + c$ /1

ii) $\int \left(\frac{7 - x^4}{x^2}\right) dx = \int (7x^{-2} - x^2)\,dx$

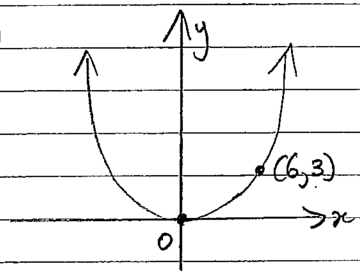
$= \frac{7x^{-1}}{-1} - \frac{x^3}{3} + c$

$= -\frac{7}{x} - \frac{x^3}{3} + c$ /2

iii) $\int (8x+3)^5\,dx = \frac{(8x+3)^6}{8 \times 6} + c$

$= \frac{(8x+3)^6}{48} + c$ /2

b)



$x^2 = 4ay$

Sub in $(6, 3)$

$36 = 4a(3)$

$36 = 12a$

$a = 3$

$\therefore x^2 = 12y$ /2

Question 2

a) $(x+4)^2 = -2(y-8)$

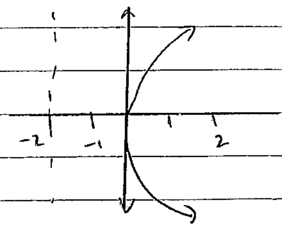
i) Vertex $(-4, 8)$ ✓

ii) focal length $4a = -2$

$a = -\frac{1}{2}$

ie $\frac{1}{2}$ unit ✓ (2)

b)



$a = 2$ ✓

$y^2 = 4ax$

$y^2 = 8x$ ✓ (2)

c)

x	$f(x)$	w	$wf(x)$
0	0	1	0
5	9	4	36
10	10	2	20
15	6	4	24
20	7	1	7

$\sum wf(x) = 87$

$A \doteqdot \frac{5}{3}(87)$

$= 145\,m^2$ (3)

1	2	3	4	5	6	7	8	9	10
1					2	3			

QUESTION 3 (7 Marks)

MARKS

a) Solve the following: $\frac{x}{x-4} \le -3$. 3

b) i. Find the points of intersection of the line $2x + y = 0$ and the parabola $y = x(x+4)$. 1

ii. Find the area bounded by $2x + y = 0$ and $y = x(x+4)$: 3

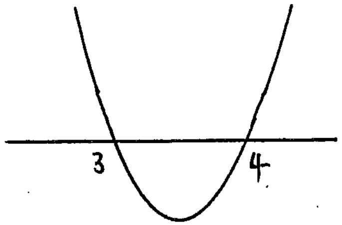
(a) $x(x-4) \le -3(x-4)^2$

$(x-4)(x + 3(x-4)) \le 0$

$(x-4)(4x-12) \le 0$

$\therefore\ 3 \le x \le 4$

but $x \ne 4$ $\quad 3 \le x < 4$

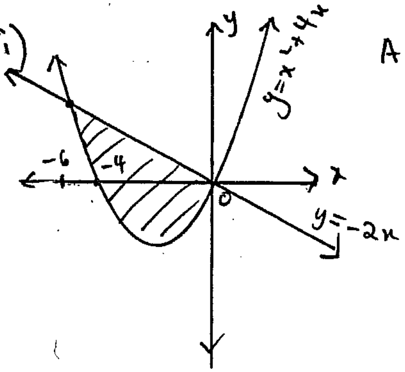


(b) (i) $2x + x(x+4) = 0$

$x^2 + 6x = 0 \qquad x(x+6) = 0$

$\therefore\ x = 0$ or $x = -6$

(ii)



$A = \int_{-6}^{0} \left(-2x - (x^2 + 4x)\right) dx$

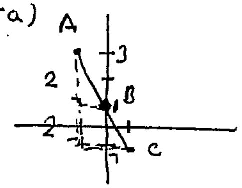
$= \int_{-6}^{0} (-x^2 - 6x)\, dx$

$= \left[-\frac{x^3}{3} - 3x^2\right]_{-6}^{0}$

$= 0 - (72 - 108)$

$A = 36$ units^2

4a)



A divides BC

externally 2:4

i.e. 1:2

b) $4a = 20$

$a = 5$

focus is $(0, 5)$

$k^2 = 20 \times 7$

$= 140$

$k = \sqrt{140}$

$PS = \sqrt{(\sqrt{140})^2 + (7-5)^2}$

$= \sqrt{140 + 4}$

$= \sqrt{144}$

$= 12$

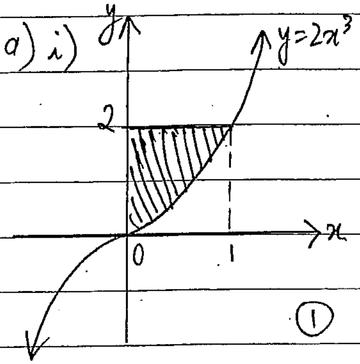
c) $\int_{-1}^{4} f(x)\, dx$

$= 5 \times 4 + \frac{1}{2} \times 3 \times 4 - \frac{1}{2} \times 4 \times 5$

$= 16$

Question 5: (7 Marks)

a) i)



(1)

ii) $V = \pi \int_0^1 y^2 \, dx$

$= \pi \int_0^1 2^2 \, dx - \pi \int_0^1 (2x^3)^2 \, dx$

$= \pi \int_0^1 (4 - 4x^6) \, dx$

$= \pi \left[4x - \frac{4x^7}{7} \right]_0^1$

$= \pi \left(4 - \frac{4}{7} \right)$

$= \frac{24\pi}{7}$ units^3. (3)

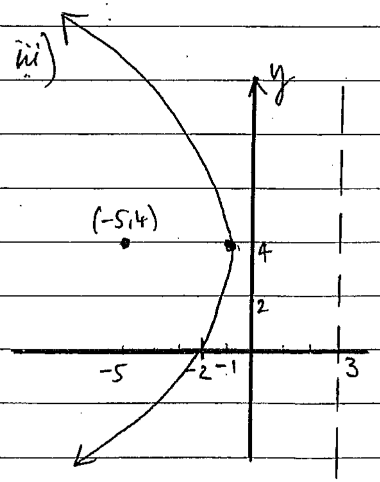
b) i) $a = 4$

$\therefore$ focus $= (-5, 4)$ (1)

ii) $(y-h)^2 = -4a(x-k)$

$\therefore (y-4)^2 = -16(x+1)$ (1)

iii)



(1)

Question 6:

a) $1^2 + 3^2 + \ldots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$

Prove true for n = 1:

$LHS = 1^2$
$= 1$

$RHS = \frac{1}{3} \times 1(2-1)(2+1)$
$= 1$

$LHS = RHS$

$\therefore$ Proven true for $n = 1$

Assume true for n = k:

$1^2 + 3^2 + \ldots + (2k-1)^2 = \frac{1}{3}k(2k-1)(2k+1)$

Prove true for n = k + 1:

R.T.P $1^2 + 3^2 + \ldots + (2k-1)^2 + (2k+1)^2 = \frac{1}{3}(k+1)(2k+1)(2k+3)$

$LHS = 1^2 + 3^2 + \ldots + (2k-1)^2 + (2k+1)^2$

$= \frac{1}{3}k(2k-1)(2k+1) + (2k+1)^2$ by assumption

$= \frac{1}{3}(2k+1)[k(2k-1) + 3(2k+1)]$

$= \frac{1}{3}(2k+1)[2k^2 + 5k + 3]$

$= \frac{1}{3}(2k+1)(k+1)(2k+3)$

$= RHS$

$\therefore$ proven true for $n = k+1$

$\therefore$ proven true by mathematical induction for $n \geq 1$

/4

b) $\int_{-\frac{8}{3}}^{-1} \frac{2}{\sqrt{1-3x}} dx = \int_{-\frac{8}{3}}^{-1} 2(1-3x)^{-\frac{1}{2}} dx$

$= 2 \left[\frac{(1-3x)^{\frac{1}{2}}}{-3 \times \frac{1}{2}} \right]_{-\frac{8}{3}}^{-1}$

$= -\frac{4}{3} \left[\sqrt{1-3x} \right]_{-\frac{8}{3}}^{-1}$

$= -\frac{4}{3} \left(\sqrt{1-3(-1)} - \sqrt{1-3\left(-\frac{8}{3}\right)} \right)$

$= -\frac{4}{3} \left(\sqrt{4} - \sqrt{9} \right)$

$= -\frac{4}{3}(2-3)$

$= \frac{4}{3}$

/3