SYDNEY GIRLS HIGH SCHOOL

YEAR 12 MATHEMATICS

ASSESSMENT TASK 2

March 2011

Time allowed: 75 minutes +5 min reading

Logs and exponentials, Trig functions I and Polynomials

Instructions:

- There are Four (4) questions. Questions **are not** of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work.
- Start each question on a new page. Write on one side of the paper only.

Name: ....

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\ x \neq 0,\ \text{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x>0$

QUESTION ONE (19 marks)

a) Change 80° into radians. (1)

b) Change $\frac{2\pi}{9}$ into degrees. (1)

c) Find the domain and the range of $y=\ln(x+3)$. (2)

d) Differentiate the following

i) $y=3\cos x$ (1)

ii) $y=e^{4x+3}$ (1)

e) Find

i) $\int 3e^{5x+6}\,dx$ (1)

ii) $\int \sin 5x\,dx$ (1)

f) Differentiate the following

i) $y=\sin x\cos x$ (2)

ii) $y=\frac{2x}{e^x+1}$ (2)

iii) $y=(e^{3x}+2)^4$ (2)

g) Find the equation of the tangent to the curve $y=e^{4x}$ which passes through the origin. (3)

h) Use Newton's method once to find an approximation for the root of x^3+x^2-1, given there is a root near $x=1$. (2)

QUESTION TWO (16 marks)

a) Find

i) $\int \frac{\sin x}{1+\cos x}\,dx$ (1)

ii) $\int \frac{e^x+e^{3x}}{e^{2x}}\,dx$ (2)

b) Find the area between the curve $y=-e^{2x}$, the x-axis and the lines $x=0$ and $x=1$. (2)

c) O is the centre of a circle with radius 10 cm. Angle $AOB=72^\circ$. A and B are points on the circumference of the circle.

i) Find the length of the arc AB (2)

ii) Find the area of the sector AOB (1)

iii) Find the area of the minor segment bounded by the chord AB (2)

d) i) State the amplitude and the period of $y=2\sin 4x$ (2)

ii) Sketch the curve in the domain $0\le x\le 2\pi$. (2)

e) The remainder when the cubic polynomial $p(x)=kx^3+4x+9$ is divided by $x-3$ is 75. Find the value of k. (2)

QUESTION THREE (20 marks)

a) Differentiate

i) $y=\ln\left(\frac{2x+1}{1-3x}\right)$ (3)

ii) $y=-4\sin^2 x$ (2)

b) The roots of the $x^3+6x^2+8x+3=0$ are α, β and γ. Find the value of

i) $\alpha+\beta+\gamma$ (1)

ii) $\alpha\beta\gamma$ (1)

iii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$ (2)

iv) $\alpha^2+\beta^2+\gamma^2$ (2)

v) $4\alpha^2\beta\gamma+4\alpha\beta^2\gamma+4\alpha\beta\gamma^2$ (2)

c) Solve $\log_e(2x+2)+\log_e x-\log_e 12=0$ (3)

d) Find the volume generated when the curve $y=\tan x$ is rotated about the x-axis between $x=0$ and $x=\frac{\pi}{4}$. (4)

QUESTION FOUR (20 marks)

a) i) Factorise the polynomial $y=2x^3-x^2-2x+1$ (3)

ii) Without using calculus sketch the curve $y=2x^3-x^2-2x+1$ (2)

iii) Hence or otherwise, find the solution to $2x^3-x^2-2x+1\le 0$ (1)

b) i) Show that $\frac{x+3}{x+5}=1-\frac{2}{x+5}$ (1)

ii) Hence find $\int\frac{x+3}{x+5}\,dx$ (2)

c) A vertical line $x=\frac{5\pi}{6}$ meets the curve $y=4\sin\left(x-\frac{\pi}{6}\right)$ at Q and $y=-3\cos\left(x+\frac{\pi}{3}\right)$ at P. Find the length of PQ in exact form. (3)

d) The roots of $x^3-15x^2-6x+k=0$ are in arithmetic progression. Find the value of k. (4)

e) When $P(x)=ax^3+bx+c$ is divided by $x-1$, the remainder is -4. When $P(x)$ is divided by x^2-4, the remainder is $-4x+3$. Find the values of a,b and c. (4)

Q1

a) $\pi = 180^\circ$

$80^\circ = \frac{\pi}{180} \times 80$

$= \frac{4\pi}{9}$

b) $\frac{2 \times 180^\circ}{9}$

$= 40$

c) Domain: $x > -3$

range: all real y

d) i) $y' = -3\sin x$

ii) $y' = 4e^{4x+3}$

e) i) $y = \frac{3e^{5x+6}}{5} + c$

ii) $y = \frac{-\cos 5x}{5} + c$

f) i) $y' = \cos x \cos x + \sin x(-\sin x)$

$= \cos^2 x - \sin^2 x$

ii) $y' = \frac{2(e^x+1) - e^x(2x)}{(e^x+1)^2}$

iii) $y' = 4(e^{3x}+2)^3 \times 3e^{3x}$

$= 12e^{3x}(e^{3x}+2)^3$

g) $y' = 4e^{4x}$

$y - 0 = 4e^{4x}(x - 0)$

$\left.\begin{array}{l} y = 4e^{4x}x \\ y = e^{4x} \end{array}\right\}$ equate to find p. of Intersec

$e^{4x} = 4e^{4x}x$

$\therefore x = \frac{1}{4}$

$\therefore y = 4e^{4 \times \frac{1}{4}} x$ $\therefore \boxed{y = 4ex}$

h) $a_1 = 1 - \frac{(1+1-1)}{3+2}$

$= 1 - \frac{1}{5}$

$= \frac{4}{5}$

Q2)

a) i) $y = \ln(1+\cos x) + C$

ii) $\int e^{-x} + e^{x}\, dx$

$= -e^{-x} + e^{x} + C$

ii) $A = \frac{1}{2}r^2\theta$

$= \frac{1}{2}(10^2)\frac{2\pi}{5}$

$= \frac{200\pi}{10}$

$= 20\pi$

b) $A = \left|\int_0^1 -e^{2x}\, dx\right|$

$= \left|\left[-\frac{e^{2x}}{2}\right]_0^1\right|$

$= \frac{-e^1}{2} + \frac{1}{2}$

$= \frac{1}{2} - \frac{e}{2}$

$= \left|\frac{1-e}{2}\right|$

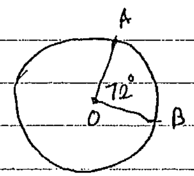
iii) $A = \frac{1}{2}r^2(\theta - \sin\theta)$

$= \frac{1}{2}100\left(\frac{2\pi}{5} - \sin\frac{2\pi}{5}\right)$

$= 50(0.3056)$

$= 15.28$

c)



i) $\ell = \pi r$

$= \frac{72\pi}{180} \times 10$

$= 4\pi$

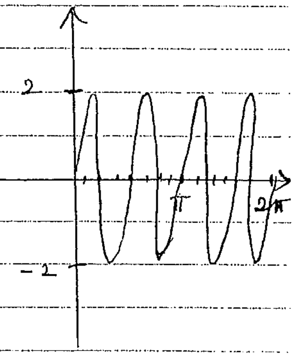
d) i) amp $= 2$

$p = \frac{2\pi}{b}$

$= \frac{2\pi}{4}$

$= \frac{\pi}{2}$

ii)



e)

$P(3) = 27k + 12 + 9$

$75 = 27k + 21$

$54 = 27k$

$\boxed{k = 2}$

Q.3) a)

i) $y = \ln(2x+1) - \ln(1-3x)$

$y' = \frac{2}{2x+1} - \frac{-3}{1-3x}$

$= \frac{2(1-3x) + 3(2x+1)}{(2x+1)(1-3x)}$

$= \frac{5}{(2x+1)(1-3x)} = \frac{5}{1-6x^2+2x}$

ii) $y = -4(\sin x)^2$

$y' = -8\sin x \cos x$

b) i) $\alpha + \beta + \gamma = -6$

ii) $\alpha\beta\gamma = -3 \qquad \alpha\beta + \beta\gamma + \gamma\alpha = 8$

iii) $\frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{8}{-3} = -2\frac{2}{3}$

iv) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$
$= (-6)^2 - 2(8)$
$= 36 - 16$
$= 20$

v) $4\alpha\beta\gamma(\alpha + \beta + \gamma)$
$= 4(-3)(-6)$
$= 72$

c) $\log_e (2x+2)x = \log_e 12$

$2x^2 + 2x = 12$

$2x^2 + 2x - 12 = 0$

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$x = -3$ or $x = 2$

$x \neq -3$

$\therefore \underline{x = 2}$

d) $V = \pi \int_0^{\frac{\pi}{4}} \tan^2 x \, dx$

$= \pi \int \sec^2 x - 1 \, dx$

$= \pi \left[\tan x - x \right]_0^{\frac{\pi}{4}}$

$= \pi \left[\tan \frac{\pi}{4} - \frac{\pi}{4} - \tan 0 - 0 \right]$

$= \pi \left[1 - \frac{\pi}{4} \right]$

$= \frac{4\pi - \pi^2}{4} \ u^3$

Q4) a)

$P(1) = 2 - 1 - 2 + 1$
$= 0$

$\therefore (x-1)$ is a factor

$$\begin{array}{r} 2x^2 + x - 1 \\ (x-1)\overline{)\,2x^3 - x^2 - 2x + 1} \\ \underline{2x^3 - 2x^2} \\ x^2 - 2x \\ \underline{x^2 - x} \\ -x + 1 \\ \underline{-x + 1} \\ 0 \end{array}$$

$2x^3 - x^2 - 2x + 1 = (x-1)(2x^2 + x - 1)$

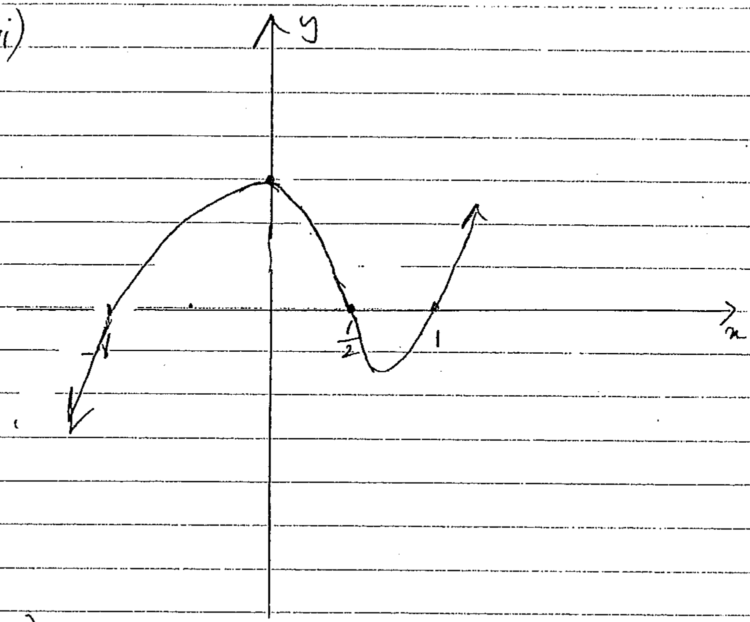
$= (x-1)(2x^2 + 2x - x - 1)$

$= (x-1)(2x(x+1) - (x+1))$

$= (x-1)(2x-1)(x+1)$

$=$

ii)



iii) $x \leq -1$, $\frac{1}{2} \leq x \leq 1$

b) i) $1 - \frac{2}{x+5}$

$= \frac{x+5-2}{x+5}$

$= \frac{x+3}{x+5}$

ii) $\int \frac{x+3}{x+5} \, dx = \int 1 - \frac{2}{x+5} \, dx$

$= x - 2 \log_e (x+5) + C$

c) $x = \frac{5\pi}{6}$

$y = 4\sin\left(\frac{4\pi}{6}\right)$

$= 4\sin\left(\frac{2\pi}{3}\right)$

$= \frac{4\sqrt{3}}{2}$

$x = \frac{5\pi}{6}$

$y = -3\cos\left(\frac{5\pi}{6} + \frac{\pi}{3}\right)$

$= -3\cos\left(\frac{7\pi}{6}\right)$

$= -3\left(\frac{-\sqrt{3}}{2}\right)$

$= \frac{3\sqrt{3}}{2}$

$PQ = \frac{\sqrt{3}}{2}$

d) $a-d,\ a,\ a+d$

$3a = 15$

$a = 5$

$(a-d)a + a(a+d) + (a+d)(a-d) = -6$

$a^2 - ad + a^2 + ad + a^2 - d^2 = -6$

$3a^2 - d^2 = -6$

$75 - d^2 = -6$

$d^2 = 81 \qquad d = \pm 9$

$d = 9 \quad a = 5$

$(a-d)(a)(a+d) = -k$

$(5-9)(5)14 = -k$

$\boxed{k = 280}$

e) $P(1) = a + b + c$

$\boxed{a + b + c = -4}$ — 1

$P(2) = 8a + 2b + c$

$-4(2) + 3 = 8a + 2b + c$

$-8 + 3 = 8a + 2b + c$

$\boxed{-5 = 8a + 2b + c}$ — 2

$P(-2) = -8a - 2b + c$

$8 + 3 = -8a - 2b + c$

$\boxed{11 = -8a - 2b + c}$ — 3

$c = -4 - a - b$ 4

$-5 = 8a + 2b - 4 - a - b$ 5

$11 = -8a - 2b - 4 - a - b$ 6

$-5 = 7a + b - 4$

$11 = -9a - 3b - 4$

$7a + b = -1$

$-9a - 3b = 15$

$21a + 3\cancel{b} = -3$

$-9a - 3b = 15$

$12a = 12$

$\boxed{a = 1}$

$7 + b = -1$

$\boxed{b = -8}$

$c = -4 - 1 + 8$

$\boxed{c = 3}$