Sydney Girls' High School

LABOR OMNIA VINCIT

2013
MATHEMATICS EXTENTION 1

YEAR 12
ASSESSMENT TASK TWO

Time Allowed: 60 minutes (plus 5 mins reading time)

TOPICS: Exponential and logarithmic Functions.
The Trigonometric Functions and Trigonometric Functions II

Directions to Candidates

- There are six (6) questions.
- Attempt ALL questions.
- Questions are of equal value.
- Start each question on a new page.
- Write on one side of the paper only.
- Show all necessary working. Marks will be deducted for careless or badly arranged work.
- Diagrams are NOT drawn to scale.
- Board-approved calculators may be used.

Total: 60 marks

NAME:.................................... TEACHER:.......................................

QUESTION 1 (10 marks) — Marks

a) Evaluate $\lim_{x\to 0}\dfrac{\sin 2x}{3x}$ — 1

b) Solve $2^{3x-1}=7$ correct to 2 decimal places — 3

c) Find $\dfrac{dy}{dx}$ if:

i) $y=e^{3x^2+5}$ — 1

ii) $y=3x\log_e x$ — 2

iii) $y=\log_e\dfrac{x^2+1}{\sqrt{x}}$ — 3

QUESTION 2 (10 marks)

a) Find: $\int xe^{x^2+1}\,dx$ — 1

b) If $\log_a 3 \doteq 1.4$ and $\log_a 2 \doteq 0.8$ evaluate $\log_a 12$ — 2

c) Find the exact volume of the solid formed by rotating the curve $y=\dfrac{1}{\sqrt{3x+1}}$ about the x-axis from $x=0$ to $x=2$. — 3

d) For the curve $y=e^x-x$

i) Find any stationary point(s) and determine their nature. — 2

ii) On a number plane sketch the graph of $y=e^x-x$ showing all relevant features. — 2

QUESTION 3 (10 marks) Marks

a) Find $\frac{dy}{dx}$ if: i) $y=\cos\left(\frac{1}{x}\right)$ 2

ii) $y=\tan^6 x$ 2

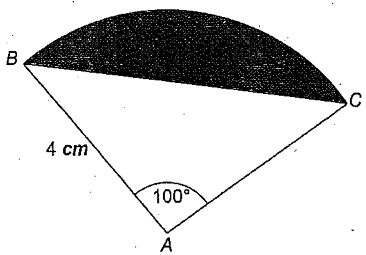
b) The area of a circle is $450cm^2$. Find in radians, the angle subtended at the centre of the circle by a $2.7cm$ arc correct to 2 decimal places. 2

c) Find $\int 3\sec^2\frac{x}{3}dx$ 1

d) Arc BC subtends an angle of 100° at the centre A of a circle with radius 4 cm.

Find the shaded area correct to 3 sig.fig 3

Figure not to scale



QUESTION 4 (10 marks) Marks

a) Sketch $y=3\sin 2x$ for $0\le x\le 2\pi$. 2

b) Given the lines $L_1: y=\frac{1}{3}x+\frac{1}{3}$ and $L_2: y=\frac{3}{4}x-3$.
Find the acute angle correct to the nearest degree formed by the two lines. 2

c) Given that $\cos 36^\circ=\frac{\sqrt{5}+1}{4}$, find in simplest form the exact value of $\cos 72^\circ$. 3

d) A curve has $\frac{d^2y}{dx^2}=18\sin 3x$ and a stationary point at $(\frac{\pi}{6},-2)$.
Find the equation of the curve. 3

QUESTION 5 (10 marks)

a) If $\sin A=\frac{1}{2}$ and $\cos B=\frac{1}{\sqrt{2}}$ where $0< A<\frac{\pi}{2}$ and $0< B<\frac{\pi}{2}$.

Find the exact value for $\cos(A+B)$. 3

b) If $\tan\frac{\theta}{2}=\frac{1}{2}$ find the exact values of $\sin 2\theta$ 3

c) i) Express $\cos\theta+\sqrt{3}\sin\theta$ in the form $R\sin(\theta+\alpha)$ where α is in radians. 2

ii) Hence or otherwise find all the values of θ in the range $0\le\theta\le 2\pi$ for which $\cos\theta+\sqrt{3}\sin\theta=1$. 2

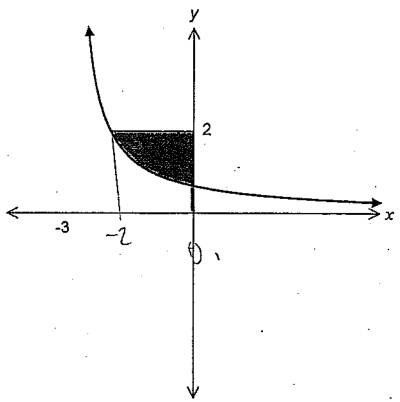
QUESTION 6 (10 marks) Marks

a) Find the indefinite integral of : $\int(\sin^2 2x)dx$ 2

b) i) Find $\frac{d}{dx}(\sin^3 3x)$ 1

ii) Hence of otherwise find $\int \sin^2 3x\cos 3x\, dx$ 1

c) Find the exact area of the region bounded by the curve $y=\frac{2}{x+3}$, the $y-axis$, and the line $y=2$. 3



d) Factorise and hence solve the following

$3\tan^3\theta+3\tan^2\theta-\tan\theta-1=0$ for $0\le\theta\le\pi$ 3

THE END

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n\neq -1;\quad x\neq 0,\text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a>0,\quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x=\log_e x,\quad x>0$

Year 12 - Task Two Ext (1)

Solutions

Question 1.

a) $\lim_{x \to 0} \frac{\sin 2x}{3x} = \frac{1}{3} \lim_{x \to 0} \frac{2 \sin 2x}{2x}$

$= \frac{2}{3}$ (1)

b) $2^{3x-1} = 7$

$\ln 2^{3x-1} = \ln 7$

$3x - 1 = \frac{\ln 7}{\ln 2}$

$3x = 2{\cdot}8 + 1$

$x = 1{\cdot}27$ (3)

c) i) $\frac{dy}{dx} = 6x e^{3x^2+5}$ (1)

ii) $y = 3x \ln x$ let $u = 3x$, $v = \ln x$

$\frac{du}{dx} = 3$ $\frac{dv}{dx} = \frac{1}{x}$

$\therefore \frac{dy}{dx} = 3 \ln x + 3x \cdot \frac{1}{x}$

$= 3 \ln x + 3$

$= 3(\ln x + 1)$ (2)

iii) $y = \ln \frac{x^2+1}{\sqrt{x}}$

$y = \ln(x^2+1) - \ln x^{1/2}$

$\frac{dy}{dx} = \frac{2x}{x^2+1} - \frac{1}{2x}$ (3)

$= \frac{4x^2 - (x^2+1)}{2x(x^2+1)} = \frac{3x^2-1}{2x(x^2+1)}$

Question 1 –

c iii) alternate solution

$y = \ln \frac{x^2+1}{\sqrt{x}}$ $\frac{u}{v}$

$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

$f'(x)$: let $u = x^2+1$ $v = x^{1/2}$

$\frac{du}{dx} = 2x$ $\frac{dv}{dx} = \frac{1}{2} x^{-1/2}$

$= \frac{1}{2\sqrt{x}}$

$\therefore f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{2x(x^{1/2}) - \frac{1}{2}x^{-1/2}(x^2+1)}{(x^{1/2})^2}$

$= \frac{2x^{3/2} - \frac{1}{2}x^{3/2} - \frac{1}{2}x^{-1/2}}{x}$

$= 2\sqrt{x} - \frac{1}{2}\sqrt{x} - \frac{1}{2}x^{-3/2}$

$f'(x) = \frac{3}{2}\sqrt{x} - \frac{1}{2}x^{-3/2}$

$\therefore \frac{dy}{dx} = \frac{f'(x)}{f(x)}$

$= \left(\frac{3}{2}\sqrt{x} - \frac{1}{2}x^{-3/2}\right) \div \left(\frac{x^2+1}{\sqrt{x}}\right)$

$\therefore \frac{dy}{dx} = \left(\frac{3}{2}\sqrt{x} - \frac{1}{2}x^{-3/2}\right) \times \frac{\sqrt{x}}{(x^2+1)}$

$= \left(\frac{3}{2}x - \frac{1}{2}x^{-1}\right) \div (x^2+1)$

$= \frac{3x^2-1}{2x} \times \frac{1}{(x^2+1)}$

$\frac{dy}{dx} = \frac{3x^2-1}{\ldots}$ (3)

Question 2

a) $\int x e^{x^2+1}\,dx = \frac{1}{2} e^{x^2+1} + C$ (1)

b) $\log_a 12 = \log_a (2 \times 2 \times 3)$

$= 2(\log_a 2) + \log_a 3$

$= 2(0.8) + 1.4$

$= 3$ (2)

c) $V = \pi \int_0^2 y^2\,dx$

$= \pi \int_0^2 \left[\frac{1}{\sqrt{3x+1}}\right]^2 dx$

$= \pi \int_0^2 \frac{1}{3x+1}\,dx$

$= \frac{1}{3} \times \pi \left[\ln(3x+1)\right]_0^2$

$= \frac{\pi}{3}\left[\ln 7 - \ln 1\right]$

Volume $= \frac{\pi}{3} \times \ln 7$ units^3 (3)

d) i) $y = e^x - x$

$\frac{dy}{dx} = e^x - 1$

For stationary pt $\frac{dy}{dx} = 0$

$e^x - 1 = 0$ at $x = 0$

$e^x = 1$ $y = e^0 - 0$ (1)

$e^x = e^0$ $y = 1$

$\therefore x = 0$ $\therefore (0,1)$ is a stationary point

Question 2 (con't)

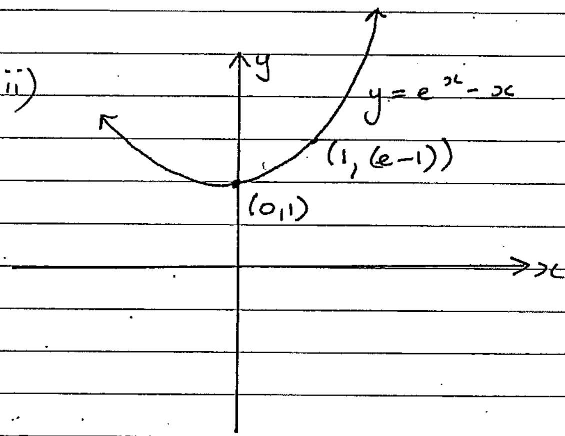
c) i) $\frac{d^2y}{dx^2} = e^x$

at $x = 0$ (1)

$\frac{d^2y}{dx^2} = 1 > 0$

$\therefore$ Minimum Value

ii)



(2)

Question 3

a) i) $y = \cos\left(\frac{1}{x}\right)$

let $u = 1/x$, $y = \cos u$

$\frac{du}{dx} = -\frac{1}{x^2}$, $\frac{dy}{du} = -\sin u$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = \frac{1}{x^2} \sin\left(\frac{1}{x}\right)$ (2)

ii) $y = (\tan x)^6$

$\frac{dy}{dx} = 6(\tan x)^5 \times \sec^2 x$

$\frac{dy}{dx} = 6\tan^5 x . \sec^2 x$ (2)

b) $A = \pi r^2$

$450 = \pi r^2$

$\therefore r = \sqrt{\frac{450}{\pi}}$

$l = r\theta$

$2 \cdot 7 = \sqrt{\frac{450}{\pi}} \times \theta$

$\therefore \theta \doteq 0 \cdot 23$ radians (2)

c) $\int 3\sec^2 \frac{x}{3}\, dx = 9\tan\frac{x}{3} + C$ (1)

d) Area of minor segment $= \frac{1}{2}r^2(\theta - \sin\theta)$

$= \frac{1}{2}(4)^2\left[\frac{5}{9}\pi - \sin 100^\circ\right]$

OR/

$= \frac{1}{2}(4)^2\left[\frac{5}{9}\pi - \sin\frac{5\pi}{9}\right]$

$= 6 \cdot 08$ cm^2

(to 3 sig. fig.) (3)

$\theta = 100^\circ$

$1^\circ = \frac{\pi}{180}$

$\therefore \theta = 100 \times \frac{\pi}{180}$

$\theta = 5/9\,\pi$

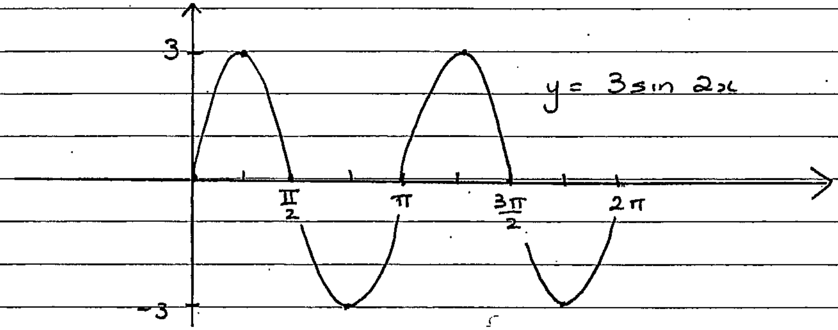
Question 4

a) Period $= \frac{2\pi}{2}$

$= \pi$

Amplitude $= 3$

Each interval $= \frac{\pi}{4}$



(2)

b) $L_1 : m_1 = 1/3$ $L_2 : m_2 = 3/4$

$\tan(\theta_2 - \theta_1) = \frac{m_2 - m_1}{1 + m_2 m_1}$

$= \frac{\frac{3}{4} - \frac{1}{3}}{1 + 3/4 . \frac{1}{3}}$

$= \frac{5/12}{5/4}$

$\tan(\theta_2 - \theta_1) = 1/3$

$\therefore \theta_2 - \theta_1 = \tan^{-1}\frac{1}{3}$ (2)

$= 18^\circ 26'$

$\doteq 18^\circ$ (nearest degree)

Question 4 (con't)

c) $\cos 72^\circ = \cos(2 \times 36^\circ)$

$$= 2\cos^2 36 - 1$$

$$= 2 \times \left(\frac{\sqrt{5}+1}{4}\right)^2 - 1$$

$$= 2\left(\frac{5+2\sqrt{5}+1}{16}\right) - 1$$

$$= \frac{6+2\sqrt{5}}{8} - \frac{8}{8}$$

$$= \frac{2\sqrt{5}-2}{8}$$

$$= \frac{\sqrt{5}-1}{4}$$

3

d) $\frac{d^2y}{dx^2} = 18 \sin 3x$

$$\int 18 \sin 3x \, dx = -6\cos 3x + C = \frac{dy}{dx}$$

at $\frac{dy}{dx} = 0$ at $x = \frac{\pi}{6}$

$$0 = -6\cos 3\left(\frac{\pi}{6}\right) + C$$

$$\therefore C = 0$$

$$\int -6\cos 3x \, dx = -2\sin 3x + C = y$$

at $x = \frac{\pi}{6}$, $y = -2$

$$-2 = -2\sin 3\left(\frac{\pi}{6}\right) + C$$

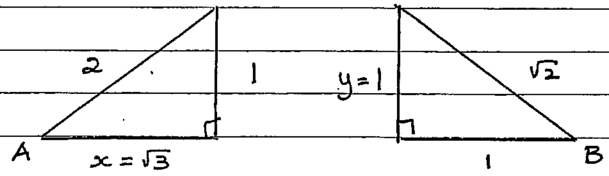
$$\therefore C = 0$$

Equation of curve $y = -2\sin 3x$

3

Question 5

a) $\sin A = \frac{1}{2}$ and $\cos B = \frac{1}{\sqrt{2}}$



$x^2 = 2^2 - 1$ $\qquad$ $y^2 = (\sqrt{2})^2 - 1$

$x = \sqrt{2}$ $\qquad$ $y = 1$

$$\therefore \cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

OR/ $= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

3

b) $\tan \theta/2 = 1/2 = t$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$= 2\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right)$$

$$= 2\left(\frac{2(\frac{1}{2})}{1+(\frac{1}{2})^2}\right)\left(\frac{1-(\frac{1}{2})^2}{1+(\frac{1}{2})^2}\right)$$

$$= 2\left(\frac{1}{5/4}\right)\left(\frac{3/4}{5/4}\right)$$

$$\sin 2\theta = \frac{24}{25}$$

3

Question 5 (con't)

c) i) $\cos\theta + \sqrt{3}\sin\theta \equiv R\sin(\theta + \alpha)$

$$R = \sqrt{1^2 + (\sqrt{3})^2}$$
$$= \sqrt{4}$$
$$\therefore R = 2$$

$$\cos\theta + \sqrt{3}\sin\theta \equiv \sin A\cos\theta + \cos A\sin\theta$$
$$\equiv 2\sin(\theta + \alpha)$$

$$\sin A = 1 \quad , \quad \cos A = \sqrt{3}$$

$$\therefore \tan A = \frac{1}{\sqrt{3}}$$
$$A = \frac{\pi}{6} = \alpha$$

(2)

$$\therefore \cos\theta + \sqrt{3}\sin\theta \equiv 2\sin(\theta + \pi/6)$$

ii) $\cos\theta + \sqrt{3}\sin\theta = 1 = 2\sin(\theta + \pi/6)$

$$\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta = \frac{1}{2} = \sin(\theta + \pi/6)$$
$$\sin(\theta + \pi/6) = 1/2$$
$$\theta + \pi/6 = \sin^{-1}\left(\tfrac{1}{2}\right)$$
$$\theta + \pi/6 = \pi/6, \quad \pi - \pi/6, \quad 2\pi + \pi/6 \ldots$$
$$\therefore \theta = (\pi/6 - \pi/6), \quad (5\pi/6 - \pi/6), \quad \frac{13\pi}{6} - \pi/6 \ldots$$
$$\theta = 0, \ 2\pi/3, \ 2\pi \quad (0 \le \theta \le 2\pi)$$

$\therefore$ Solution: $\theta = 0$, $2\pi/3$ and 2π

(2)

Question 6

$$= \tfrac{1}{2}\int (1 - \cos 2ax)\,d$$

a) $\int \sin^2 2x \, dx$: NB: $\int \sin^2 ax \, dx = \frac{x}{2} - \frac{1}{4a}\sin 2ax$

$$\int \sin^2 x \, dx = \frac{1}{2}\int 1 - \cos 2x \, dx$$
$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

$$\therefore \int \sin^2 2x \, dx = \frac{1}{2}x - \frac{1}{8}\sin 4x + C$$

(2)

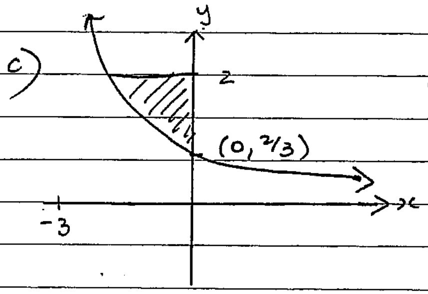
b) i) $\frac{d}{dx}(\sin^3 3x) = 3(\sin 3x)^2 \times 3\cos 3x$
$$= 9\sin^2 3x \cos 3x$$

(1)

ii) $\int \sin^2 3x \cos 3x \, dx = \frac{1}{9}\sin^3 3x + C$

(1)

c)



$$y = \frac{2}{x+3}$$

at $x = 0$, $y = 2/3$

$$x + 3 = \frac{2}{y}$$
$$x = 2/y - 3$$

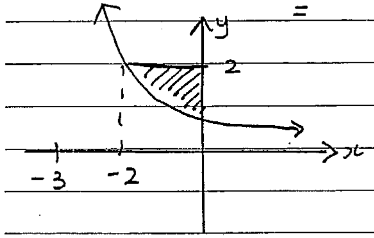
$$\text{Area} = \left| \int_{2/3}^{2} \frac{2}{y} - 3 \, dy \right|$$
$$= \left| \Big[2\ln y - 3y \Big]_{2/3}^{2} \right|$$

(3)

$$= \left| \left[(2\ln 2 - 6) - (2\ln 2/3 - 2) \right] \right|$$
$$= \left| \left[2\ln 2 - (2\ln 2 - 2\ln 3) - 4 \right] \right|$$
$$= |2\ln 3 - 4| \text{ units}^2$$

Question 6

OR/ Area = Area of square $- \int_{-2}^{0} \frac{2}{x+3}\,dx$



$= 2^2 - \int_{-2}^{0} \frac{2}{x+3}\,dx$

$= 4 - 2\left[\ln(x+3)\right]_{-2}^{0}$

$= 4 - 2\left[\ln 3 - \ln 1\right]$

$= 4 - 2\ln 3$ (exact value.)

$\doteqdot 1.8$

$y = \frac{2}{x+3}$

at $y = 2$

$2 = \frac{2}{x+3}$

$x + 3 = 1$

$x = -2$

3

d) $3\tan^3\theta + 3\tan^2\theta - \tan\theta - 1 = 0$

let $x = \tan\theta$ $\quad 0 \leq \theta \leq \pi$

$3x^3 + 3x^2 - x - 1 = 0$

$3x^2(x+1) - 1(x+1) = 0$

$(x+1)(3x^2 - 1) = 0$

$\therefore (\tan\theta + 1)(3\tan^2\theta - 1) = 0$

$\tan\theta = -1$ OR $3\tan^2\theta = 1$

S	A
T	C

$\theta = \frac{3\pi}{4}$

$\tan^2\theta = \frac{1}{3}$

$\tan\theta = \pm\frac{1}{\sqrt{3}}$

$\theta = \pi/6, 5\pi/6$

3

$\therefore \theta = \pi/6, 3\pi/4, 5\pi/6.$