Sydney Girls High School

2013

YEAR 12 HSC ASSESSMENT TASK 2

MATHEMATICS EXTENSION 2

Time Allowed: 60 minutes + 5 minutes reading time

Topic: Complex Numbers

Total: 60 marks

General Instructions:

- There are THREE (3) Questions which are of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- Start each Question on a new page.
- Write on one side of the paper only.
- Diagrams are NOT to scale.
- Board-approved calculators may be used.
- Write your name clearly at the top of each question and clearly number each question.

Student Name :________________ **Teacher Name :**________________

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 **20 Marks**

a) If z is the complex number $-2+2\sqrt{3}i$, find:

i) $|z|$ **1**

ii) $\arg z$ **1**

iii) Indicate on an Argand diagram, the complex numbers: **4**

$z,\ -z,\ iz,\ \bar{z}$ and $\dfrac{1}{z}$

iv) Show that $z^2=4\bar{z}$ **2**

v) Show that z is a root of the equation $w^3-64=0$ and find the other roots. **2**

b) Simplify i^{2013} . **1**

c) $1,\ \omega,\ \omega^2$ are the three cube roots of unity.

i) Simplify $\dfrac{1-\omega-\omega^2}{1-\omega+\omega^2}$. Give your answer as a simple fraction. **2**

ii) Form a quadratic equation with integer coefficients whose roots are $2-\omega$ and $2-\omega^2$. **1**

d) If ω is a non-real complex nth root of unity such that **2**

$z^n-1=(z-1)(z-\omega)(z-\omega^2)\times....\times(z-\omega^{n-1})$, show that $1+\omega+\omega^2+....+\omega^{n-1}=0$.

e)

i) Find the Cartesian equation of the locus of z if $|z-i|=\operatorname{Im}(z)$. **2**

ii) Find the gradients of the tangents to this curve, which pass through the origin. Hence find the set of possible values of $\arg z$, where $-\pi<\arg z\le\pi$. **2**

Question 2 (Start a New Page) **20 Marks**

a) Given $z_1=i\sqrt{2}$ and $z_2=\dfrac{2}{1-i}$:

i) Express z_1 and z_2 in modulus-argument form. **2**

ii) On an Argand diagram plot the points P and R representing the complex numbers z_1 and z_2 respectively and the point Q representing z_1+z_2 . **2**

iii) Giving reasons, show $\arg(z_1+z_2)=\dfrac{3\pi}{8}$. **2**

iv) Use the diagram to find the exact value of $\tan\dfrac{3\pi}{8}$ **1**

b) If z is a complex number, mark on a separate Argand diagram the regions of the complex plane, satisfied by:

i) $\arg(z+1)\le\dfrac{\pi}{4}$ **2**

ii) $1\le|z-1|<2$ **2**

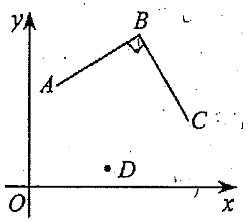
c)

i) Find the roots of the equation $z^5+1=0$ in modulus-argument form **2**

ii) Factorise $z^5+1=0$ into real linear and quadratic factors. **2**

iii) Deduce that $\cos\dfrac{\pi}{5}+\cos\dfrac{3\pi}{5}=\dfrac{1}{2}$ and $\cos\dfrac{\pi}{5}\cos\dfrac{3\pi}{5}=-\dfrac{1}{4}$. **2**

iv) Write a quadratic equation with integer coefficients which has roots $\cos\dfrac{\pi}{5}$ and $\cos\dfrac{3\pi}{5}$. Hence find the exact values of $\cos\dfrac{\pi}{5}$ and $\cos\dfrac{3\pi}{5}$. **3** a)



In the diagram the vertices of ΔABC are represented by the complex numbers z_1, z_2 and z_3 respectively. The triangle is isosceles and right-angled at B.

i) Explain why $(z_3 - z_2)^2 = -(z_1 - z_2)^2$. **2**

ii) D is the point such that $ABCD$ is a square. Find the complex number, in terms of z_1, z_2 and z_3, that represents D. **1**

b)

i) If $z = \cos\theta + i\sin\theta$, prove that $z^n + z^{-n} = 2\cos n\theta$ **1**

ii) Hence solve the equation $2z^4 + 3z^3 + 5z^2 + 3z + 2 = 0$, expressing the roots in the form $a + ib$. **4**

c)

i) Use De Moivre's Theorem to show $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **2**

ii) Deduce $8x^3 - 6x - 1 = 0$ has solutions $x = \cos\theta$ when $\cos 3\theta = \frac{1}{2}$. **1**

iii) Find the roots of $8x^3 - 6x - 1 = 0$ in the form $x = \cos\theta$. **2**

iv) Hence evaluate, without the use of a calculator $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$. **2**

d)

i) Sketch the locus of a point z which moves such that: **2**

$$\arg\left(\frac{z-i}{z+i}\right) = \frac{\pi}{3}$$

ii) Find the Cartesian equation of the locus. **3**

End of paper

SGHS MATHEMATICS EXTENSION 2

2013 ~ TASK 2 ~ SOLUTIONS

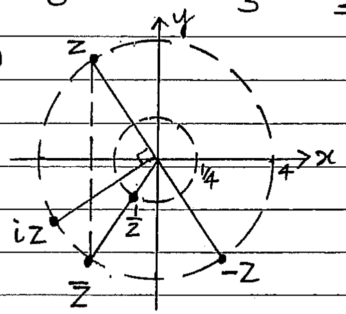
Question 1

a) $z = -2 + 2\sqrt{3}\,i$

i) $|z| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4$ ①

ii) $\arg z = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ ①

iii)



④

iv) $z = -2 + 2\sqrt{3}\,i$

$z^2 = (-2 + 2\sqrt{3}\,i)^2$

$= 4 - 4\sqrt{3}\,i - 4\sqrt{3}\,i + 4\times 3 i^2$

$= (4-12) - 8\sqrt{3}\,i$

$= -8 - 8\sqrt{3}\,i$

$4\bar{z} = 4(-2 + 2\sqrt{3}\,i)$

$= -8 - 8\sqrt{3}\,i$

$\therefore z^2 = 4\bar{z}$ ②

OR $z^2 = \left[4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)\right]^2$

$= 16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$4\bar{z} = 4\left[4\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right)\right]$

$= 16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$\therefore z^2 = 4\bar{z}$

1 a) v) $\omega^3 - 64 = 0$

$(\omega - 4)(\omega^2 + 4\omega + 16) = 0$

$\therefore \omega = 4$ or $\omega = \frac{-4 \pm \sqrt{16-64}}{2}$

$= \frac{-4 \pm 4\sqrt{3}\,i}{2}$

$= -2 \pm 2\sqrt{3}\,i$

$\therefore$ roots are 4, z and $\bar{z}$. ②

OR $\omega^3 - 64 = 0$

$\omega^3 = 64$

$(r\operatorname{cis}\theta)^3 = 64\operatorname{cis}0$

$r^3\operatorname{cis}3\theta = 64\operatorname{cis}0$

$r^3 = 64$

$r = 4$

$3\theta = 0 + 2n\pi$

$\theta = \frac{2n\pi}{3}$

$n = 0,\ \theta = 0$

$n = 1,\ \theta = \frac{2\pi}{3}$

$n = -1,\ \theta = \frac{-2\pi}{3}$

$\therefore$ Roots are: $z_1 = 4\operatorname{cis}0 = 4$

$z_2 = 4\operatorname{cis}\frac{2\pi}{3}$

$= 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= 4\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right)$

$= -2 + 2\sqrt{3}\,i$

$= z$, as required

$z_3 = 4\operatorname{cis}\left(\frac{-2\pi}{3}\right)$

$= -2 - 2\sqrt{3}\,i$

$= \bar{z}$.

Q1 b) $i^{2013} = i \cdot i^{2012}$

$= i \cdot (i^4)^{503}$

$= i \cdot 1$

$= i$ ①

Q1 c) i) $\dfrac{1-\omega-\omega^2}{1-\omega+\omega^2}$ $\left[\begin{array}{r} 1+\omega+\omega^2=0 \\ \omega^3=1 \end{array}\right]$

$= \dfrac{1-\omega-(-1-\omega)}{1-\omega-1-\omega}$

$= \dfrac{1-\omega+1+\omega}{-2\omega}$

$= \dfrac{2}{-2\omega}$

$= -\dfrac{1}{\omega}$ ②

ii) sum of roots $= 2-\omega+2-\omega^2$

$= 2+2-(\omega+\omega^2)$

$= 2+2+1$

$= 5$

product of roots $= (2-\omega)(2-\omega^2)$

$= 4-2\omega^2-2\omega+\omega^3$

$= 4-2(\omega^2+\omega)+1$

$= 4+2+1$

$= 7$

$\therefore$ Quadratic Equation: $x^2-5x+7=0$ ①.

Q1 d) Method 1:

- $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are the roots of the equation $z^n-1=0$.
- sum of the roots $= -\dfrac{b}{a} = \dfrac{-\text{coeff } z^{n-1}}{\text{coeff } z^n} = -\dfrac{0}{1} = 0$

$\therefore 1+\omega+\omega^2+\ldots+\omega^{n-1}=0$.

Method 2:

- $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are the roots of $z^n-1=0$
- If ω is a non-real zero, then $\omega^n-1=0$

$\omega^n=1$.

- $1+\omega+\omega^2+\ldots+\omega^{n-1}$ is the sum of a G.P.

where $a=1$, $r=\omega$.

Sum of roots $= \dfrac{a(1-r^n)}{1-r}$

$= \dfrac{1(1-\omega^n)}{1-\omega}$

$= \dfrac{0}{1}$ $[\omega^n=1,\ \omega\neq 1]$

$= 0$

$\therefore 1+\omega+\omega^2+\ldots+\omega^{n-1}=0$.

Method 3:

- Factorising: $z^n-1=(z-1)(z^{n-1}+z^{n-2}+\ldots+z^2+z+1)$
- Since ω is a non-real zero of z^n-1

then $\omega^n-1=0$.

That is: $(\omega-1)(\omega^{n-1}+\omega^{n-2}+\ldots+\omega^2+\omega+1)=0$

- But $\omega\neq 1$

$\therefore 1+\omega+\omega^2+\ldots+\omega^{n-1}=0$.

②

Q1 e) i) $|z-i| = \operatorname{Im} z$

$$|x+i(y-1)| = y$$

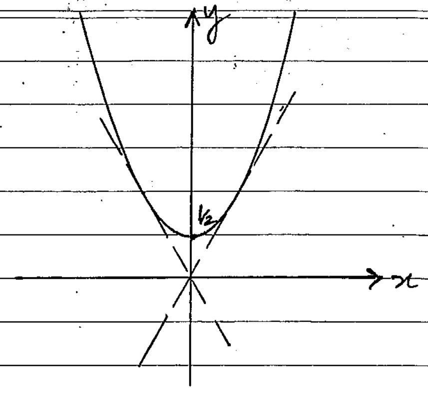
$$\sqrt{x^2+(y-1)^2} = y$$

$$x^2+(y-1)^2 = y^2$$

$$x^2+\cancel{y^2}-2y+1 = \cancel{y^2}$$

$$x^2+1 = 2y$$

$$y = \tfrac{1}{2}(x^2+1) \quad \therefore \text{ a parabola.} \quad (2)$$



ii)

- $y=mx$ is a tangent to $y=\frac{1}{2}(x^2+1)$, passing through the origin.
- For points of contact solve:

$$mx = \tfrac{1}{2}(x^2+1)$$

$$2mx = x^2+1$$

$$\therefore x^2-2mx+1=0$$

- Equal roots when $\Delta = 0$.

$$b^2-4ac=0$$

$$4m^2-4 = 0$$

$$4(m-1)(m+1)=0$$

$$\therefore m = +1, -1.$$

$\therefore$ tangents are: $y=x$ or $y=-x$.

$$\therefore \tan^{-1}(1) \le \arg z \le \tan^{-1}(-1)$$

$$\frac{\pi}{4} \le \arg z \le \frac{3\pi}{4} \quad (2)$$

Question 2

a)

i) $z_1 = i\sqrt{2}$

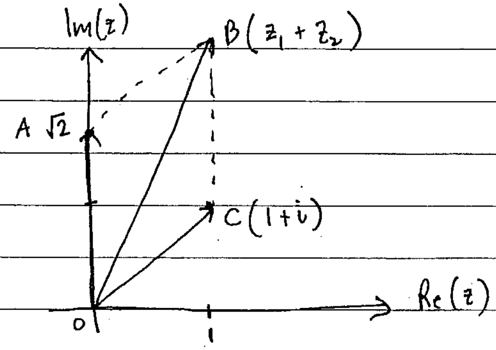
$= \sqrt{2}\operatorname{cis}\pi/2$

$z_2 = \dfrac{2}{1-i} \times \dfrac{1+i}{1+i}$

$= 1+i$

$= \sqrt{2}\operatorname{cis}\pi/4$

ii)



iii) $\angle AOC = \pi/2 - \pi/4$

$= \pi/4$

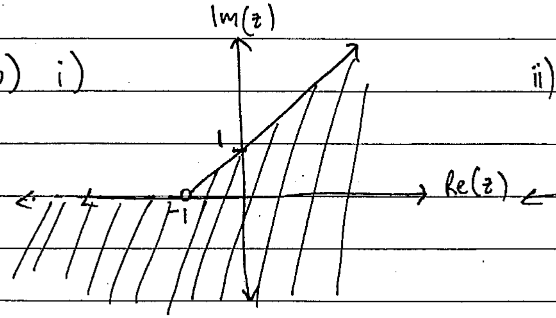
$\angle BOC = \pi/4 \div 2$ (Diagonals of a rhombus bisect $\angle$s through which they pass)

$\therefore \operatorname{Arg}(z_1+z_2) = \pi/4 + \pi/2$

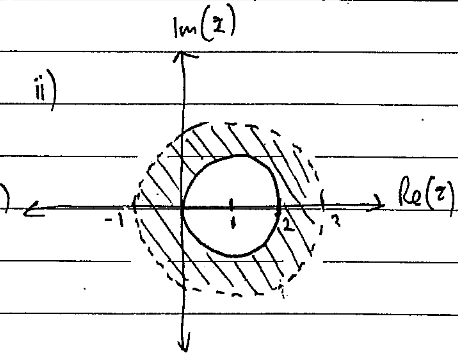
$= 3\pi/8$

iv) $\tan 3\pi/8 = \sqrt{2}+1$

b) i)



ii)



c) i) $z^5 = -1$

let $z = r\,\text{cis}\,\theta$

$r^5 \text{cis}\,5\theta = \text{cis}(\pi + 2k\pi) \quad k = 0, 1, 2, 3, 4$

$r^5 = 1$

$r = 1$

$\text{cis}\,5\theta = \text{cis}(\pi + 2k\pi)$

$5\theta = \pi + 2k\pi$

$\theta = \dfrac{\pi + 2k\pi}{5}$

$= \pi/5,\ 3\pi/5,\ \pi,\ -3\pi/5,\ -\pi/5$

$z = \text{cis}\,\pi/5,\ \text{cis}\,3\pi/5,\ -1,\ \text{cis}\,{-3\pi/5},\ \text{cis}\,{-\pi/5}$

ii) $z^5 + 1 = (z+1)(z^2 - 2\cos\pi/5\, z + 1)(z^2 - 2\cos 3\pi/5\, z + 1)$

iii) Equating coefficient of z:

$0 = 1 - 2\cos\pi/5 - 2\cos 3\pi/5$

$1/2 = \cos\pi/5 + \cos 3\pi/5$

Equating coefficient of z^3:

$0 = 4\cos\pi/5\cos 3\pi/5 + 1 - 2\cos 3\pi/5 - 2\cos\pi/5 + 1$

$0 = 4\cos\pi/5\cos 3\pi/5 + 1$

$-1/4 = \cos\pi/5\cos 3\pi/5$

iv) $x^2 - 1/2\,x - 1/4 = 0 \qquad \therefore \cos\pi/5 = \dfrac{1+\sqrt5}{4}$

$4x^2 - 2x - 1 = 0$

$x = \dfrac{2 \pm \sqrt{4+16}}{8} \qquad \cos 3\pi/5 = \dfrac{1-\sqrt5}{4}$

$= \dfrac{1 \pm \sqrt5}{4}$

[illegible]

$z_3 = i(z_1 - z_2) + z_2$

$(z_3 - z_2)^2 = i^2(z_1 - z_2)^2$

$= -(z_1 - z_2)^2$

ii) $z_1 - z_2 + z_3 - z_2 + z_2$

$= z_1 + z_3 - z_2$

b) i) $(\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin)^{-n}$

$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

ii) $2z^2 + 3z + 5 + \frac{3}{z} + \frac{2}{z^2} = 0$

$2\left(z^2 + \frac{1}{z^2}\right) + 3\left(z + \frac{1}{z}\right) + 5 = 0$

$2 \times 2\cos 2\theta + 3 \times 2\cos\theta + 5 = 0$

$4\cos 2\theta + 6\cos\theta + 5 = 0$

$4(2\cos^2\theta - 1) + 6\cos\theta + 5 = 0$

$8\cos^2\theta + 6\cos\theta + 1 = 0$

$8\cos^2\theta + 4\cos\theta + 2\cos\theta + 1 = 0$

$4\cos\theta(2\cos\theta + 1) + 1(2\cos\theta + 1) = 0$

$(4\cos\theta + 1)(2\cos\theta + 1) = 0$

$\cos\theta = -\frac14,\ -\frac12$

$\sin\theta = \pm\sqrt{1 - (\frac14)^2}$ or $\sin\theta = \pm\sqrt{1 - (\frac12)^2}$

$= \pm\frac{\sqrt{15}}{4} \qquad = \pm\frac{\sqrt3}{2}$

$\therefore z = -\frac14 \pm \frac{\sqrt{15}}{4}i,\ -\frac12 \pm \frac{\sqrt3}{2}i$

c) i) $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos(1 - \cos^2\theta)$

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

ii) $\frac12 = 4x^3 - 3x$

$1 = 8x^3 - 6x$

$8x^3 - 6x - 1 = 0$

iii) $\cos 3\theta = \frac12$

$3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}, \frac{17\pi}{3}$

$\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{17\pi}{9}$

$x_1 = \cos\frac{\pi}{9} = \cos\frac{17\pi}{9}$

$x_2 = \cos\frac{5\pi}{9} = \cos\frac{13\pi}{9}$

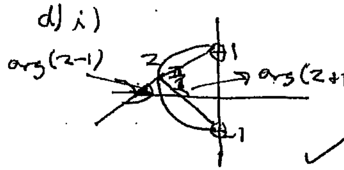
$x_3 = \cos\frac{7\pi}{9} = \cos\frac{11\pi}{9}$

iv) $\cos\frac{\pi}{9} \times \cos\frac{7\pi}{9} \times \cos\frac{5\pi}{9}$

$= -\frac{-1}{8}$

$= \frac18$

d) i)



$\arg(z+1) - \arg(z-1) + \frac{\pi}{3} = 2\pi$

$= \frac{5\pi}{3}$

$\arg(z-1) - \arg(z+1) = -\frac{5\pi}{3}$

$\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$

(ii) $\frac{z-i}{z+i} = \frac{x+iy-i}{x+iy+i} \times \frac{x-i(y+1)}{x-i(y+1)}$

$= \frac{(x-i)^2 + y^2}{x^2 + (y+1)^2}$

$= \frac{x^2 + y^2 - 1 - 2xi}{x^2 + (y+1)^2}$

$\frac{\text{Im}\left(\frac{z-i}{z+i}\right)}{\text{Re}\left(\frac{z-i}{z+i}\right)} = \tan\frac{\pi}{3}$

$\frac{-2x}{x^2 + y^2 - 1} = \sqrt3$

$-\frac{2x}{\sqrt3} = x^2 + y^2 - 1$

$1 + (\frac{1}{\sqrt3})^2 = x^2 + 2x + (\frac{1}{\sqrt3})^2 + y^2$

$\therefore (x + \frac{1}{\sqrt3})^2 + y^2 = \frac43$

$-2x > 0$

$x > 0$