Sydney Girls High School

2012
Mathematics Extension 1

YEAR 12
ASSESSMENT TASK 3

General Instructions:

- Reading time – 5 minutes
- Working time – 60 minutes
- Total marks - 55
- Attempt Questions 1-5. All questions are of equal value.
- Show all necessary working. Marks may be deducted for badly arranged work or incomplete working.
- Write on one side of the paper only. Start each question on a new page.
- Write your student number clearly at the top of each question and clearly number each question.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.

Topics: Integration by Substitution, Inverse Functions & Inverse Trigonometric Functions, The Parabola & Parametrics, Circle Geometry, Mathematical Induction (Divisibility & Inequalities), Polynomials

Student Name :____________ **Teacher Name :**____________

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 (11 Marks)

MARKS

(a) Given $f(x)=x^3+2$, find $f^{-1}(x)$. 2

(b) Find $\displaystyle\int \frac{8x^3}{\left(1+x^4\right)^2}\,dx$ using the substitution $u=1+x^4$. 3

(c) Differentiate :

(i) $y=\cos^{-1}(5x)$ 1

(ii) $y=\tan^{-1}\left(x^3\right)$ 2

(d) Find :

(i) $\displaystyle\int \frac{dx}{36+x^2}$ 1

(ii) $\displaystyle\int \frac{3}{\sqrt{9-4x^2}}\,dx$ 2

End of Question 1

Question 2 (11 Marks)

MARKS

(a) $P(x)=x^3+ax^2+bx+6$ is a polynomial with a factor of $(x-3)$. When $P(x)$ is divided by $(x+2)$, the remainder is 40. Find the values of a and b. 4

(b) Evaluate $\displaystyle\int_{2\sqrt{2}}^{4} \sqrt{16-x^2}\,dx$ using the substitution $x=4\sin\theta$. Give your answer in simplest exact form. 4

(c) Sketch the graph of $y=3\sin^{-1}\left(\dfrac{x}{2}\right)$, clearly indicating the domain and range. 3

End of Question 2

Question 3 (11 Marks)

MARKS

(a) Tangents are drawn to the parabola $x^2 = 12y$ from an external point $(4,-1)$. Find the equation of the chord of contact, expressing your answer in general form. 2

(b) Sketch the graph of $f(x) = (1-x^2)(x^2-x)$, showing all intercepts. 3

(c) The function $f(x) = x\tan^{-1}(x) - 2$ has a zero near $x = 1.5$.
Use one application of Newton's method to obtain another approximation to this zero. Give your answer correct to 3 decimal places. 3

(d) Find the general solution to the equation $2\sin^2 x + \cos x = 1$. 3

End of Question 3

Question 4 (11 Marks)

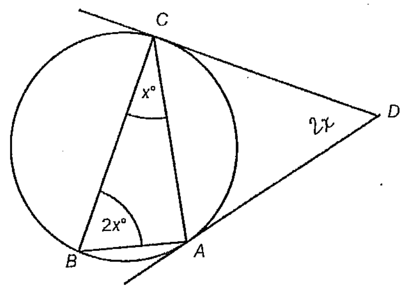
MARKS

(a) (i) Find the gradient of the tangent to the parabola $x^2 = 4ay$ at the point $P(2ap, ap^2)$. 1

(ii) OQ is parallel to the tangent at P, where Q has coordinates $(2aq, aq^2)$ and O is the origin. Show that $q = 2p$. 1

(iii) If M is the midpoint of PQ, find the equation of the locus of M as P and Q vary such that OQ remains parallel to the tangent at P. 3

(b) AD and CD are tangents to a circle. B is a point on the circle such that $\angle CBA$ and $\angle CDA$ are equal and are both double $\angle BCA$. Prove that BC is a diameter of the circle.
(Diagram NOT to scale.) 4



(c) Expand $\sin\left(2\cos^{-1}\left(\frac{1}{x}\right)\right)$ and express your answer in simplest exact form. 2

End of Question 4

Question 5 (11 Marks)

MARKS

(a) Prove by mathematical induction that $9^{n+2} - 4^n$ is divisible by 5 for all positive integers n. **4**

(b) Given α, β and γ are the roots of $2x^3 - 13x^2 - x + 3 = 0$, find the value of:

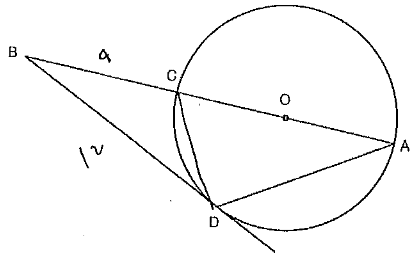
(i) $\alpha + \beta + \gamma$ **1**

(ii) $\alpha\beta\gamma$ **1**

(iii) $\left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\gamma}\right)\left(\gamma + \frac{1}{\alpha}\right)$ **2**

(c) O is the centre of the circle and BD is a tangent to the circle at D. Find the size of $\angle DAB$ correct to the nearest minute given BC is 9 cm and BD is 12 cm. **2**

(Diagram NOT to scale.)



(d) State the domain for the function $f(x) = \cos^{-1}\left(\sqrt{\tan^{-1} x}\right)$. **1**

End of paper

Question 1

a) Given $f(x) = x^3 + 2$ find

$f^{-1}(x)$

let $y = x^3 + 2$

$\therefore \; x = y^3 + 2$

$x - 2 = y^3$

$\therefore \; \sqrt[3]{x-2} = y$

$\therefore \; f^{-1}(x) = \sqrt[3]{x-2}$ (2)

b) $\displaystyle\int \frac{8x^3}{(1+x^4)^2}\,dx$

let $u = 1 + x^4$

$\dfrac{du}{dx} = 4x^3$

$du = 4x^3\,dx$

$\displaystyle\therefore \int \frac{8x^3}{(1+x^4)^2}\,dx = \int \frac{2\,du}{u^2}$

$\displaystyle = 2\int u^{-2}\,du$

$= 2\dfrac{u^{-1}}{-1} + C$

$= \dfrac{-2}{1+x^4} + C$ (3)

c) i) $y = \cos^{-1} 5x$

$\dfrac{dy}{dx} = \dfrac{-5}{\sqrt{1-25x^2}}$ (1)

ii) $y = \tan^{-1} x^3$

$\dfrac{dy}{dx} = \dfrac{3x^2}{1+(x^3)^2}$

$= \dfrac{3x^2}{1+x^6}$ (2)

d) i) $\displaystyle\int \frac{dx}{36+x^2}$

$= \frac{1}{6}\tan^{-1}\frac{x}{6} +$ (1)

ii) $\displaystyle\int \frac{3}{\sqrt{9-4x^2}}\,dx$

$\displaystyle = 3\int \frac{1}{\sqrt{9-4x^2}}\,dx$

$= 3 \times \frac{1}{2}\sin^{-1}\frac{2x}{3} + C$

$= \frac{3}{2}\sin^{-1}\frac{2x}{3} + C$ (2)

Total Marks = 11

Ext 1 Q2 2012 Task 3

a) $P(3) = 27 + 9a + 3b + 6$

$0 = 33 + 9a + 3b$

$P(-2) = -8 + 4a - 2b + 6$

$40 = -2 + 4a - 2b$

$9a + 3b = -33$

$4a - 2b = 42$

$18a + 6b = -66$

$12a - 6b = 126$

$30a = 60$

$\boxed{a = 2}$

$8 - 2b = 42$

$-2b = 34$

$\boxed{b = -17}$

b) $x = 4\sin\theta$ $\qquad 4 = 4\sin\theta$

$\frac{dx}{d\theta} = 4\cos\theta$ $\qquad \sin\theta = 1$

$\qquad \theta = \frac{\pi}{2}$

$dx = 4\cos\theta\, d\theta$

$\qquad 2\sqrt{2} = 4\sin\theta$

$\int \sqrt{16 - 16\sin^2\theta} \cdot 4\cos\theta\, d\theta$ $\qquad \sin\theta = \frac{\sqrt{2}}{2}$

$= \int 4\sqrt{1 - \sin^2\theta} \cdot 4\cos\theta\, d\theta$ $\qquad \theta = \frac{\pi}{4}$

$= \int 4\cos\theta \cdot 4\cos\theta\, d\theta$

$= \int 16\cos^2\theta\, d\theta$

$16\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2} + \frac{\cos 2\theta}{2}\, d\theta$

$= 16\left[\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right]$

$= 16\left[\left[\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{\sin\pi}{4}\right] - \left[\frac{1}{2}\left(\frac{\pi}{4}\right) + \frac{\sin\frac{\pi}{2}}{4}\right]\right]$

$= 16\left[\left[\frac{\pi}{4} + 0\right] - \left[\frac{\pi}{8} + \frac{1}{4}\right]\right]$

$= 16\left(\frac{\pi}{4} - \frac{\pi}{8} - \frac{1}{4}\right)$

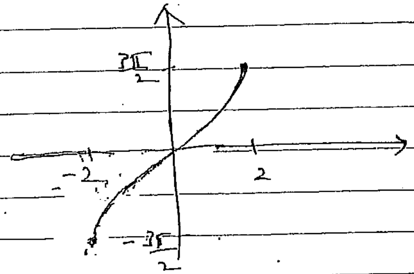
$= 16\left(\frac{\pi}{8} - \frac{1}{4}\right)$

$= 2\pi - 4$

c) Domain

$-2 \le x \le 2$

$-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$



$x^2 = 12y$

$4a = 12$

$a = 3$ 1

$xx_1 = 2a(y + y_1)$

$4x = 2\times 3(y-1)$

$4x = 6y - 6$

$4x - 6y + 6 = 0$

$2x - 3y + 3 = 0$ 1

$f(x) = (1-x^2)(x^2 - x)$ take $\ominus$ out of this one $= (x-1)$.

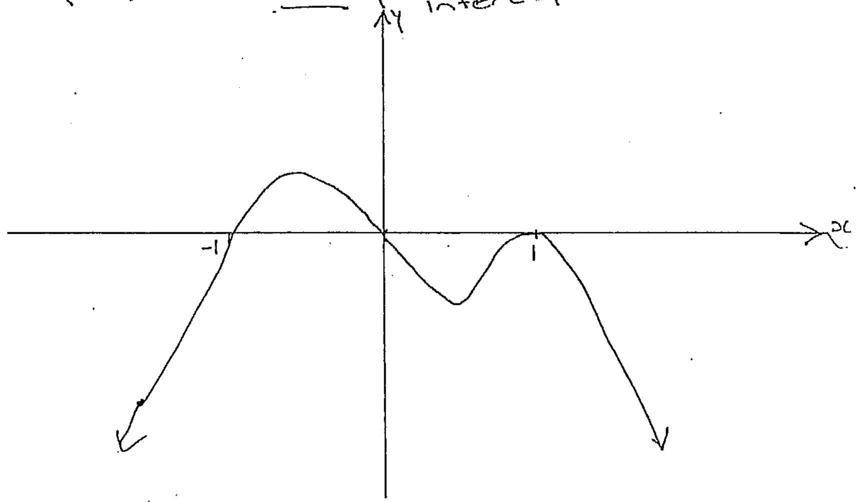
$= x(x-1)(1+x)(1-x)$

$= -x(x-1)^2(1+x)$

1 eqn

1 shape

1 intercepts



$f'(x) = x \cdot \dfrac{1}{1+x^2} + \tan^{-1} x$

$= \dfrac{x}{1+x^2} + \tan^{-1} x$ — 1

$f'(1.5) = \dfrac{1.5}{1+(1.5)^2} + \tan^{-1}(1.5)$

$= 1.444 \ldots$ — 1

$f(1.5) = 1.5\tan^{-1}(1.5) - 2$

$= -0.5258 \ldots$

$x_2 = 1.5 - \dfrac{f(1.5)}{f'(1.5)}$ — 1

$= 1.86405 \ldots$

$= 1.864$ (3 dec. pl)

d) $2\sin^2 x + \cos x = 1$

$2(1-\cos^2 x) + \cos x = 1$

$2 - 2\cos^2 x + \cos x = 1$

$2\cos^2 x - \cos x - 1 = 0$ — 1

$(2\cos x + 1)(\cos x - 1) = 0$

$2\cos x + 1 = 0$

$\cos x = -\frac{1}{2}$

$x = 2n\pi \pm \left(\frac{2\pi}{3}\right)$ — 1

$\cos x - 1 = 0$

$\cos x = 1$

$x = 2n\pi$ — 1

Extn 1 Task 3 2012

Question 4

a) i) $x = 2ap \quad , \quad y = ap^2$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{x}{2a}$

at $x = 2ap$

$\frac{dy}{dx} = \frac{2ap}{2a}$

$= p$ (1)

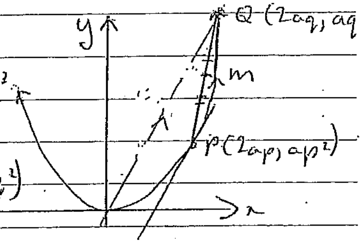
ii) $m_{OQ} = \frac{aq^2 - 0}{2aq - 0}$

$= \frac{q}{2}$

ie $p = \frac{q}{2}$ (1)

$2p = q$

iii) Co-ords of M



$x = \frac{2ap + 2aq}{2}, \quad y = \frac{ap^2 + aq^2}{2}$

$x = a(p+q), \quad y = \frac{a}{2}(p^2 + q^2)$ ✓

Now $q = 2p$

$\therefore x = a(3p), \quad y = \frac{a}{2}(5p^2)$ ✓

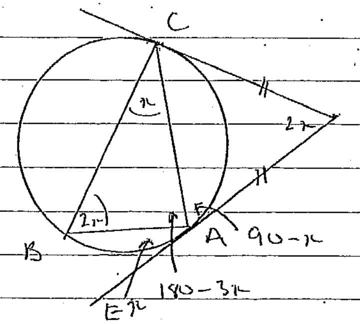
$\therefore p = \frac{x}{3a}$ sub in

$y = \frac{a}{2}\left(5\left(\frac{x^2}{9a^2}\right)\right)$ ✓

$= \frac{a}{2} \times \frac{5x^2}{9a^2}$

$y = \frac{5x^2}{18a}$ (3)

b) (long solution)



$CD = AD$ (tangent from ext)

$\therefore \angle BCA = \angle DAC$

$= \frac{180° - 2x}{2}$ (∠ sum

$= 90° - x$ ✓

Also $\angle BAC = 180° - 3x$ (∠ sum Δ) ✓

Now $\angle BAE = \angle BCA$ (∠ in alt segment)

$= x$ ✓

Then $\angle DAC + \angle BAC + \angle BAE = 180°$ (straight line)

$90 - x + 180 - 3x + x = 180°$

$-3x + 270° = 180°$

$x = 30°$ (4)

But $\angle CAB = 180 - 3x$

$= 90°$

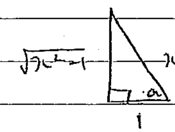
$\therefore$ BC a diameter (∠ in semi circle)

c) Let $a = \cos^{-1}\frac{1}{x}$

then $\cos a = \frac{1}{x}$ and $\sin(2\cos^{-1}\frac{1}{x})$

$= \sin 2a$

$= 2\sin a \cos a$



$\therefore \sin(2\cos^{-1}\frac{1}{x}) = 2\,\frac{\sqrt{x^2-1}}{x} \times \frac{1}{x}$

$= 2\,\frac{\sqrt{x^2-1}}{x^2}$ (2)

1	2	3	4	5	6	7	8	9	10
7	~~8~~ 5		4		~~3~~ 3	6	2	1	

Question 5 (11 MARKS)

(a) Step 1: Prove true for $n=1$

$$9^{1+2} - 4^{1} = 725$$
$$= 5 \times 145$$ 1

$\therefore$ Divisible by 5 for $n=1$

Step 2: Assume true for $n=k$.

i.e. assume $9^{k+2} - 4^{k} = 5A \Rightarrow 9^{k+2} = 5A + 4^{k}$
where A is some integer

Step 3: Prove true for $n=k+1$.

i.e. prove $9^{k+1+2} - 4^{k+1} = 5B$
(B is some integer)

$$\begin{aligned}
LHS &= 9^{1} \times 9^{k+2} - 4^{1} \times 4^{k} \quad 1 \\
&= 9(5A + 4^{k}) - 4 \times 4^{k} \quad 1 \\
&= 45A + 9 \times 4^{k} - 4 \times 4^{k} \\
&= 45A + 5 \times 4^{k} \\
&= 5(9A + 4^{k}) \\
&= 5B \quad \text{where } B = 9A + 4^{k} \quad 1
\end{aligned}$$

$\therefore LHS = RHS$

Step 4: If true for $n=k$, proven true for $n=k+1$. Since proven true for $n=1$, must be true for $n=2$. Since true for $n=2$, true for $n=3$ and so on. Hence, true for all positive integers.

Question 5 (continued)

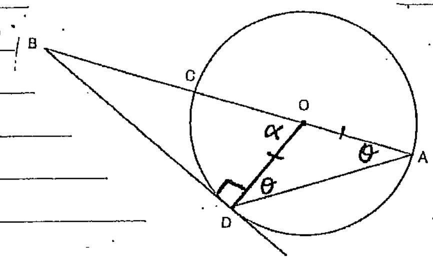
(b) (i) $\alpha + \beta + \gamma = -\frac{b}{a} = \frac{13}{2}$ 1

(ii) $\alpha\beta\gamma = -\frac{d}{a} = -\frac{3}{2}$ 1

(iii) $\left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\gamma}\right)\left(\gamma + \frac{1}{\alpha}\right)$

$$\begin{aligned}
&= \left(\alpha\beta + \frac{\alpha}{\gamma} + 1 + \frac{1}{\beta\gamma}\right)\left(\gamma + \frac{1}{\alpha}\right) \\
&= \alpha\beta\gamma + \beta + \alpha + \frac{1}{\gamma} + \gamma + \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\alpha\beta\gamma} \quad 1 \\
&= -\frac{3}{2} + \frac{13}{2} - \frac{2}{3} + \frac{\alpha\beta + \alpha\gamma + \beta\gamma}{\alpha\beta\gamma} \\
&= \frac{13}{3} + \frac{\left(-\frac{1}{2}\right)}{\left(-\frac{3}{2}\right)} \quad 1 \\
&= \frac{14}{3}
\end{aligned}$$

(c)



let d = diameter

$$\begin{aligned}
9 \times (9 + d) &= 12^{2} \\
d + 9 &= 16 \\
\therefore d &= 7 \quad 1 \\
\tan\alpha &= \frac{12}{3.5} \\
\alpha &= 73°44'
\end{aligned}$$

$\theta + \theta = \alpha \Rightarrow \theta = \frac{\alpha}{2} = 36°52'$ 1

(d) $-1 \le \sqrt{\tan^{-1}x} \le 1 \Rightarrow 0 \le \sqrt{\tan^{-1}x} \le 1$

$$0 \le \tan^{-1}x \le 1$$

$\therefore \quad 0 \le x \le \tan 1$ 1

(Note: $\tan 1 \doteqdot 1.557$)