SYDNEY GIRLS HIGH SCHOOL

LABOR OMNIA VINCIT

YEAR 12 MATHEMATICS
Extension 1
2014

ASSESSMENT TASK 3

June 12, 2014

Time allowed: 60 minutes +5 min reading

Inverse Functions, Integration by substitution, Circle geometry and Parametrics

Instructions:

- There are Seven (7) questions. Questions **are not** of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work.
- Start each question on a new page. Write on one side of the paper only.

Name: ....

Teacher's Name ....

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}+C, \quad n\neq -1;\quad x\neq 0 \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x + C, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}+C, \quad a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C, \quad a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C, \quad a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C, \quad a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax + C, \quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C, \quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right) + C, \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right) + C$$

NOTE: $\ln x = \log_e x,\ x>0$

Question Three (7 mark)

a) Differentiate $y=(\tan^{-1}x)^2$ (1)

b) The point $P(2at,at^2)$ is on the parabola $x^2=4ay$. The normal to the parabola at P cuts the x-axis at X and the y-axis at Y.

i) Find the equation of the normal at P. (2)

ii) Find the coordinates of X and Y. (2)

iii) Find the value of t such that P is the midpoint of XY. (2)

Question Four (7 marks)

a) Find the general solution (in radians) of the equation $\sqrt{3}\sin x=\cos x$. (2)

b) Find the area enclosed between the curve $y=\dfrac{9}{\sqrt{9-x^2}}$, the x-axis, the lines $x=0$ and $x=\dfrac{3}{2}$. (2)

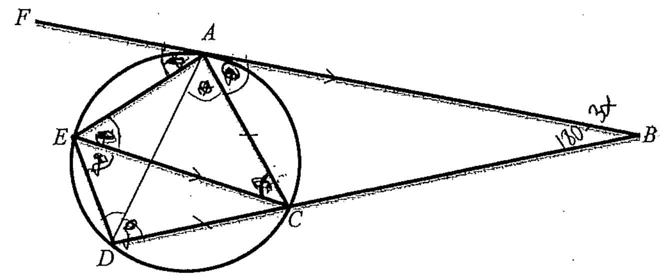
c) Find the exact value of $\sin\left[\cos^{-1}\dfrac{4}{5}+\tan^{-1}\left(\dfrac{-4}{3}\right)\right]$ (3)

Question Seven (7 marks)

a) AB is a tangent to the circle. $AB \parallel EC$ and $CD=AC$.

i) Copy the diagram on to your answer sheet.

ii) Prove that $AC \parallel ED$. (4)



b) Find the derivative of the following function in simplest form ,where $x>0$.

$y=\sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ (3)

Question Five (7 marks)

a) Point $P(2t,t^2)$ is on the parabola $x^2=4y$. The tangent to the parabola at point P cuts the x-axis at T. M is the midpoint of PT.

i) Find the equation of the tangent at P. (1)

ii) Show that M has coordinates $\left(\frac{3t}{2},\frac{t^2}{2}\right)$. (2)

iii) Hence, find the Cartesian equation of the locus of M as P moves on the parabola. (2)

b) Consider the function $f(x)=-3\sin^{-1}(x+1)$.

Sketch $y=f(x)$, giving the coordinates of its endpoints and any intercepts with the coordinate axis. (2)

Question Six (7 marks)

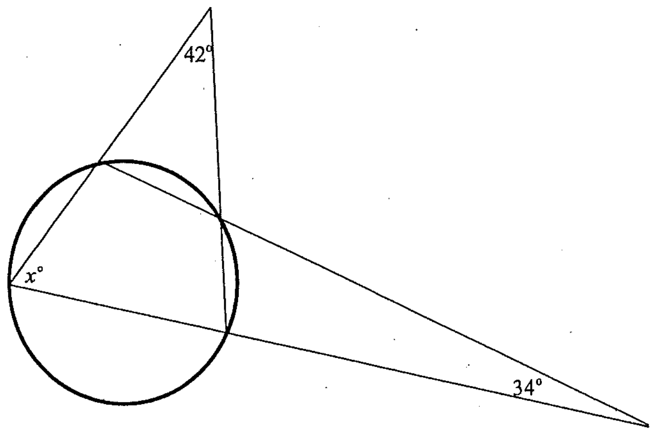
a) A function is defined as $f(x)=1+e^{2x}$

i) Find the inverse function $f^{-1}(x)$ in terms of x. (2)

ii) State the domain and range of the inverse function. (2)

iii) Sketch $f(x)$ and $f^{-1}(x)$ on the same diagram. (1)

b) Copy the diagram on your answer sheet and find the value of x. (2)



Question one (7 marks)

a) Evaluate $\int_0^3 \frac{3x}{1+x}\,dx$ using substitution $u=1+x$ (3)

b) Integrate $\int \frac{dx}{\sqrt{9-4x^2}}$ (2)

c) Determine the exact value of $\cos\left(\tan^{-1}\left(\frac{8}{11}\right)\right)$. (2)

Question two (8 marks)

a) Find $\int \frac{\sin 2x}{1+\sin^2 x}\,dx$, using the substitution $u=\sin^2 x$. (3)

b) The region bounded by the curve $y=\frac{1}{\sqrt{3+x^2}}$ and the x-axis between the lines $x=1$ and $x=3$ is rotated about the x- axis. Find the exact volume of the solid formed. (3)

c) Differentiate $\cos^{-1}\left(-\frac{1}{x}\right)$ with respect to x. Answer in simplified form. (2)

Sydney Girls High School

Mathematics Faculty

Years 12 HSC Mathematics Extension 1

2014 Task 3

Question	Marker's Comment
1	This question was done well. Some students couldn't integrate $\frac{1}{u}$ in part a.
2	a) Many students lost marks in this question as they confused inverse trigonometric integrals with integrals of logarithms. b) Almost, all students did very well in this question. c) Many students forgot how to differentiate an inverse trigonometric function.
3	This question was done very well. Some students in b) iii) didn't realise midpoint of XY = P and so, $(\frac{2at+at^3}{2},\frac{2a+at^2}{2})=2at,at^2$ which allowed us to solve for $t=\pm\sqrt{2}$. Full marks were awarded for $t=\sqrt{2}$.
4	Most students gained full marks for this question. Part c) Some students were confused whether to use sum or difference with two angles and so were not able to obtain the right answer which was -7/25.
5	This question was well done by the vast majority of students.
6	Most students gained high marks in this question. a) i) In terms of x means that y must be the subject of the equation. ii) The domain was $x>1$ not $x\geq 1$ as $x=1$ is an asymptote of the inverse function. iii) This part could be marked correct from previous answer (CFPA). b) There where many different solutions, the best ones could be done in two or three lines.
7	(a) A number of proofs were convoluted and difficult to follow. The best solutions introduced a single pronumeral and proceeded to prove that $\angle DEC$ and $\angle ACE$ were both equal to this pronumeral. A general note that some students should improve their use of reasons. (b) Common errors were misunderstanding of the derivative of $\sin^{-1}\left[f(x)\right]$, failure to use the chain rule (and the quotient rule) and inability to simplify.

Yr 12 Ext 1 2014 Task 3

1) a

$u = 1 + x$

$\frac{du}{dx} = 1$

$3 \int \frac{u-1}{u}\, du$

$3 \int_1^4 1 - \frac{1}{u}\, du$

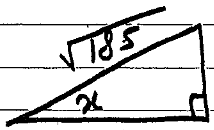
$3 \left[u - \ln u \right]_1^4$

$3 \left[4 - \ln 4 - 1 + \ln 1 \right]$

$= 3 \left[3 - \ln 4 \right]$

b) $\int \frac{dx}{2\sqrt{\frac{9}{4} - x^2}}$

$= \frac{1}{2} \sin^{-1} \frac{2x}{3} + c$

c) 

$\cos x = \frac{11}{\sqrt{185}}$

Q2 a) $\int \frac{\sin 2x}{1 + \sin^2 x}\, dx$ Let $u = \sin^2 x$

$\frac{du}{dx} = 2 \sin x \cos x$ ✓

$du = \sin 2x\, dx$

$I = \int \frac{du}{1+u} = \ln(1+u) + c$ ✓

$= \ln(1 + \sin^2 x) + c$ ✓

b) $V = \pi \int_a^b y^2\, dx$

$V = \pi \int_1^3 \frac{1}{3 + x^2}\, dx$ ✓

$V = \pi \int_1^3 \frac{1}{(\sqrt{3})^2 + x^2}\, dx$

$= \frac{\pi}{\sqrt{3}} \left[\tan^{-1} \frac{x}{\sqrt{3}} \right]_1^3$ ✓

$= \frac{\pi}{\sqrt{3}} \left[\frac{\pi}{3} - \frac{\pi}{6} \right] = \frac{\pi^2}{6\sqrt{3}}$ ✓ units^3

Q3 c) $y = \cos^{-1}\left(-\frac{1}{x}\right)$

$y' = \frac{-f'(x)}{\sqrt{1 - [f(x)]^2}}$

$y' = \frac{-1/x^2}{\sqrt{1 - \frac{1}{x^2}}}$ ✓

$y' = \frac{-1/x^2}{\sqrt{\frac{x^2-1}{x^2}}}$

$= \frac{-1/x^2}{\frac{\sqrt{x^2-1}}{x}} = \frac{-1}{x\sqrt{x^2-1}}$ ✓

Question 3 - Ext 1

a) $y = (\tan^{-1} x)^2$

let $u = \tan^{-1} x$ $\quad y = u^2$

$\frac{du}{dx} = \frac{1}{1+x^2}$ $\quad \frac{dy}{du} = 2u$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 2u \times \frac{1}{1+x^2}$

$= \frac{2(\tan^{-1}x)}{1+x^2}$ 1

b) i) $x^2 = 4ay$

$\frac{x^2}{4a} = y$

$\therefore \frac{dy}{dx} = \frac{2x}{4a}$

$= \frac{x}{2a}$

at $P(2at, at^2)$

$m = \frac{2at}{2a}$

$m_1 = t$

$\therefore m_2 = -\frac{1}{t}$

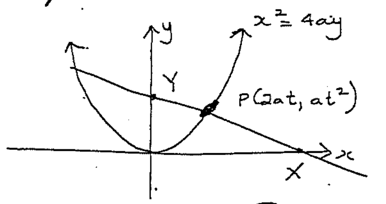
Equation of normal

$y - y_1 = m(x - x_1)$

$y - at^2 = -\frac{1}{t}(x - 2at)$

$y = -\frac{1}{t}x + 2a + at^2$

OR $x + ty - 2at - at^3$



2

ii) Co-ordinates of X and Y

$y = -\frac{1}{t}x + 2a + at^2$

at $x = 0$, $y = 2a + at^2$

$\therefore Y(0, 2a + at^2)$

at $y = 0$

$0 = -\frac{1}{t}x + 2a + at$

$-2a - at^2 = -\frac{1}{t}x$

$2at + at^3 = x$

$\therefore X(2at + at^3, 0)$ 2

iii) Midpoint of $XY = P$

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = (2at, at^2)$

$\left(\frac{0 + 2at + at^3}{2}, \frac{2a + at^2 + 0}{2}\right) = 2at, at^2$

$\frac{2at + at^3}{2} = 2at$ and $\frac{2a + at^2}{2} = at^2$

$\cancel{a}\cancel{t}\left(\frac{2+t^2}{2}\right) = 2\cancel{a}\cancel{t}$ OR $\cancel{a}\left(\frac{2+t^2}{2}\right) = \cancel{a}t^2$

$2 + t^2 = 4$

$t^2 = 2$

$t = \pm\sqrt{2}$

$\frac{2+t^2}{2} = t^2$

$2 + t^2 = 2t^2$

$2 = t^2$

$\therefore t = \pm\sqrt{2}$

2

7

Question 4 (7 marks).

a) $\sqrt{3}\sin x = \cos x$.

$\frac{\sin x}{\cos x} = \frac{1}{\sqrt{3}}$

$\tan x = \frac{1}{\sqrt{3}}$

$x = n\pi + \frac{\pi}{6}$. .. 2

b) $\text{Area} = \int_0^{3/2} \frac{9}{\sqrt{9-x^2}}\,dx$

$= 9\int_0^{3/2} \frac{1}{\sqrt{9-x^2}}$

$= 9\left[\sin^{-1}\frac{x}{3}\right]_0^{3/2}$.

$= 9\left[\sin^{-1}\frac{1}{3}\cdot\frac{3}{2} - \sin^{-1}0\right]$

$= 9\left[\frac{\pi}{6} - 0\right]$

$= \frac{3\pi}{2}$ units^2. 2

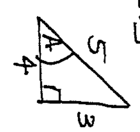
c) $\sin\left[\cos^{-1}\frac{4}{5} + \tan^{-1}\left(-\frac{4}{3}\right)\right]$

$= \sin\left[\cos^{-1}\frac{4}{5} - \tan^{-1}\left(\frac{4}{3}\right)\right]$

Let $\cos^{-1}\frac{4}{5} = A$

$\therefore \cos A = \frac{4}{5}$

$\therefore \sin A = \frac{3}{5}$

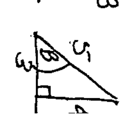


Let $\tan^{-1}\frac{4}{3} = B$

$\therefore \tan B = \frac{4}{3}$

$\sin B = \frac{4}{5}$

$\cos B = \frac{3}{5}$.



$= \sin(A - B)$

$= \sin A\cos B - \cos A\sin B$.

$= \left(\frac{3}{5}\right)\left(\frac{3}{5}\right) - \left(\frac{4}{5}\right)\left(\frac{4}{5}\right)$ 3

$= \frac{9}{25} - \frac{16}{25}$

$= -\frac{7}{25}$. .

[illegible] ✓

ii) At T $y = 0$

$-t^2 = tx - 2t^2$

$t^2 = tx$

$x = t$

$M = \left(\frac{2t+t}{2}, \frac{t^2+0}{2}\right)$

$= \left(\frac{3t}{2}, \frac{t^2}{2}\right)$ ✓✓

iii) $x = \frac{3t}{2}$

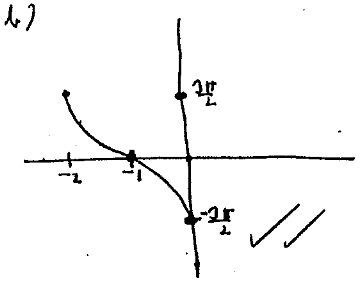
$t = \frac{2x}{3}$

$y = \frac{\left(\frac{2x}{3}\right)^2}{2}$

$= \frac{4x^2}{18}$

$= \frac{2x^2}{9}$ ✓✓

b)



✓✓

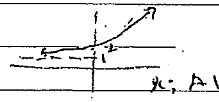
Question Six

i) $x = 1 + e^{2y}$

$x - 1 = e^{2y}$

$\log_e(x-1) = 2y$

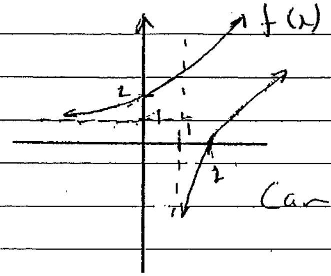
$y = \frac{1}{2}\log_e(x-1)$ (2)



ii) $x > 1$ ✓

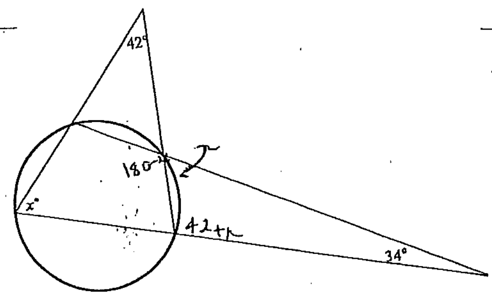
y: all real ✓ (2)

iii)



(1)

Can be CFPA



$x + (42 + x) + 34° = 180°$

$2x = 104°$

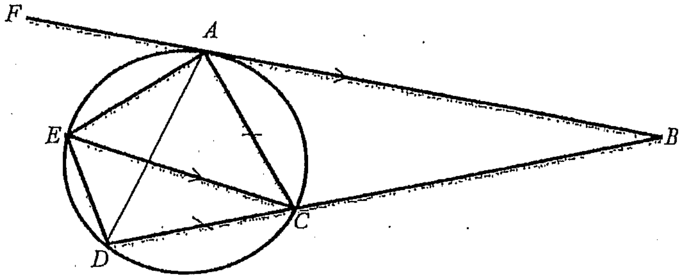
$x = 52°$ (2)

Question 7 (7 marks)

a) AB is a tangent to the circle. $AB \parallel EC$ and $CD = AC$.

i) Copy the diagram on to your answer sheet.

ii) Prove that $AC \parallel ED$. (4)



Let $\angle BAC = x$.

$\angle ADC = x$ ($\angle$ in alt. segment)

ΔACD is isosceles ($AC = DC$, given)

$\therefore \angle DAC = \angle ADC = x$ (base $\angle$s of isos. Δ)

$\angle DEC = \angle DAC = x$ ($\angle$s in same segment)

$\angle ACE = \angle BAC = x$ (alt. $\angle$s, $AB \parallel EC$)

$\therefore \angle DEC = \angle ACE$ (both equal x)

$\therefore AC \parallel ED$ (alt. $\angle$s only equal on parallel lines)

b) Find the derivative of the following function in simplest form, where $x > 0$.

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) \qquad (3)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \times \frac{(1+x^2)\times -2x - (1-x^2)\times 2x}{(1+x^2)^2}$$

$$= \frac{-2x-2x^3-2x+2x^3}{(1+x^2)\sqrt{(1+x^2)^2\left(1-\frac{(1-x^2)^2}{(1+x^2)^2}\right)}}$$

$$= \frac{-4x}{(1+x^2)\sqrt{(1+x^2)^2-(1-x^2)^2}}$$

$$= \frac{-4x}{(1+x^2)\sqrt{1+2x^2+x^4-(1-2x^2+x^4)}}$$

$$= \frac{-4x}{(1+x^2)\sqrt{4x^2}}$$

$$= \frac{-4x}{2x(1+x^2)} \quad \text{since } x>0$$

$$= \frac{-2}{1+x^2}$$