SYDNEY GIRLS HIGH SCHOOL

LABOR OMNIA VINCIT

MATHEMATICS

HSC Assessment Task 3

2012

Reading time: 5 minutes

Time allowed: 90 minutes

Topics: Locus, Integration and Exponential functions

DIRECTIONS TO CANDIDATES:

- Attempt all questions
- Questions are of equal value
- There are 4 questions with part marks shown
- All necessary working must be shown. Marks may be deducted for careless or badly arranged work
- Board approved calculators may be used
- Write on one side of the paper only

STUDENT NUMBER/NAME: ....

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1 *(20 Marks)*

a) Evaluate $\frac{e^3-5}{\sqrt[3]{e^2}}$ correct to three significant figures. 2

b) For the parabola $x^2=12(y+1)$ find :

i) the coordinates of the vertex 1

ii) the equation of the directrix 2

c) Differentiate with respect to x: $e^{3x^2}+\frac{1}{e^x}$ 2

d) Integrate:

i) $\int (x^2-2x)\,dx$ 2

ii) $\int (e^x-2)(e^x+2)\,dx$ 2

e) Find $\int_{-1}^{0}(2x+1)^4\,dx$ 2

f) Find the equation of the tangent to $x^2=8y$ at the point $P(4,2)$ 3

g) Find:

i) $\int x^2\sqrt{4x}\,dx$ 2

ii) $\int \frac{1-e^x}{e^x}\,dx$ 2

Question 2 *(20 Marks)*

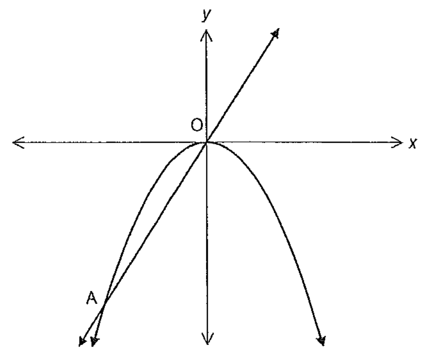
a) Given that $y=3e^{2x}$ find the value of $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y$ 3

b) The table shows the values of a function $f(x)$ for five values of x 3

x	0	1	2	3	4
$f(x)$	2	3	12	35	80

Use Simpson's rule with these five values to find an approximation to $\int_0^4 f(x)\,dx$

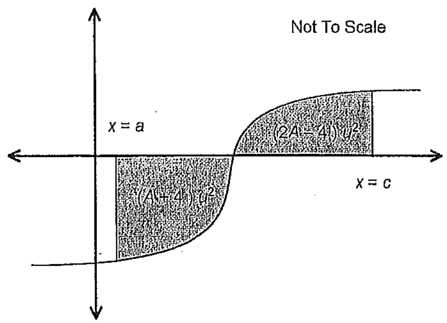
c) The curve $y=-x^2$ and the line $y=3x$ intersect at point O and A as shown in the diagram.



i) Find the coordinates of A 2

ii) Find the area bounded by the curve $y=-x^2$ and the line $y=3x$. 3

d) In the diagram, the area between the curve $y=f(x)$, the x-axis and the line $x=a$ is equal $(A+4)u^2$. The area between the curve, the x-axis and the line $x=c$ is equal to $(2A-4)u^2$. Use this information to evaluate $\int_a^c f(x)\,dx$. 2



e) Find the centre and radius for the circle whose equation is $x^2-2x+y^2+4y-11=0$. 3

f) A point $P(x,y)$ moves so that it is twice the distance from the line $x=1$ as it is from the line $y=2$. Find the equation of the locus of the point P. 4

Question 3 *(20 Marks)*

a) A cone is generated by rotating the line $y=3x$ about the *x-axis* from *x = 0* to *x = 3*.

i) State the integral required to evaluate the volume of the cone. 1

ii) Using the trapezoidal rule with four function values, approximate the volume of the cone. 2

iii) Calculate the exact volume of the cone and hence find the percentage error in the approximation. 3

b) Consider the function $f(x)=4xe^{-2x}$

i) State the y-intercept of the function 1

ii) Find $f'(x)$ 2

iii) State the coordinates of the stationary point, and determine the nature of the stationary point. 2

iv) Find the coordinates of any points of inflexion. 3

v) Sketch the graph of $y=f(x)$ 2

c) The point $P\left(2,\, e^{6}\right)$ lies on the curve $y=e^{3x}$.

i) Find the equation of the normal to the curve at P. 2

ii) Find the coordinates of T where the normal meets the x-axis. 2

Question 4 *(20 Marks)*

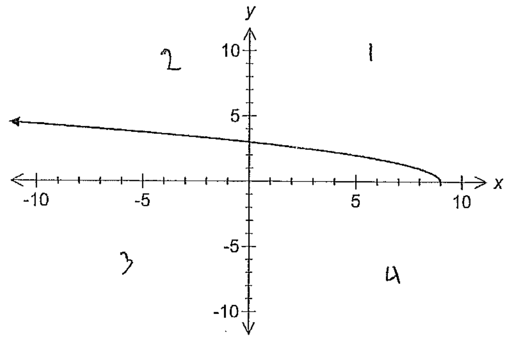
a) A parabola has equation $y^2-4y-4x-8=0$.

i) Find the coordinates of the vertex. 2

ii) Sketch the parabola, showing the location of the vertex, focus and directrix. 2

b) Derive the equation of the locus of the point $P(x,y)$ that moves so that it is equidistant from the point $(a,0)$ and the line $x+a=0$ 4

c) The mould for a wine glass is obtained by rotating that part of the curve $y=\sqrt{9-x}$ which lies in the first quadrant through one complete revolution about the *x-axis*.



i) Find the volume of the mould if the measurements are in centimetres. 3

ii) Find the number of glasses of wine that can be obtained from a one litre bottle if each glass is two-thirds full. 1

d) The area under the curve $y=1-e^{-x}$, above the *x-axis* and between $x=0$ and $x=1$, is rotated about the *x-axis*. Find the exact volume of the solid generated. 4

e) Find the exact volume of the solid generated when the area between the curve $y=x^2+2$, the *x* and *y* axes and the line $x=4$ is rotated about the y-axis. 4

HSC Assessment task 3

2012

Mathematics

Question 1

a) $\frac{e^3-5}{\sqrt[3]{e^2}} = 7.74517\ldots$

$= 7.75$

b) $x^2 = 12(y+1)$

i) $V = (0,-1)$

ii) $a = 3$

$\therefore$ The equation of the directrix is $y = -4$

c) $y = e^{3x^2} + \frac{1}{e^x}$

$y = e^{3x^2} + e^{-x}$

$y' = 6x\,e^{3x^2} - e^{-x}$

$y' = 6x\,e^{3x^2} - \frac{1}{e^x}$

d) i) $\int (x^2 - 2x)\,dx$

$= \frac{x^3}{3} - \frac{2x^2}{2} + c$

$= \frac{x^3}{3} - x^2 + c$

g) i) $\int x^2\sqrt{4x}\,dx$

$= \int x^2 \times 4^{1/2} x^{1/2}\,dx$

$= \int 2\,x^{\frac{5}{2}}\,dx$

$= \frac{2\,x^{7/2}}{7/2} + c$

$= \frac{4\,x^{7/2}}{7} + c$

$= \frac{4}{7}\sqrt{x^7} + c$

ii) $\int \frac{1-e^x}{e^x}\,dx$

$= \int \left(\frac{1}{e^x} - \frac{e^x}{e^x}\right) dx$

$= \int (e^{-x} - 1)\,dx$

$= -e^{-x} - x + c$

d) ii) $\int (e^x - 2)(e^x + 2)\, dx$

$= \int ((e^x)^2 - 4)\, dx$

$= \int (e^{2x} - 4)\, dx$

$= \frac{1}{2} e^{2x} - 4x + c$

e) $\int_{-1}^{0} (2x+1)^4\, dx$

$= \left[\frac{(2x+1)^5}{2 \times 5} \right]_{-1}^{0}$

$= \left[\frac{1}{10} - \frac{-1}{10} \right]$

$= \frac{2}{10}$

$= \frac{1}{5}$

f) $x^2 = 8y$

$y = \frac{x^2}{8}$

$\frac{dy}{dx} = \frac{2x}{8}$

$= \frac{x}{4}$

at $x = 4$

The gradient of the tangent is

$m = \frac{4}{4}$

$m = 1$

The equation of the tangent is

$y - 2 = 1(x - 4)$

$\therefore y = x - 2$

or $x - y - 2 = 0$

Question 2

20 marks

a) $y = 3e^{2x}$

$\frac{dy}{dx} = 6e^{2x}$

$\frac{d^2y}{dx^2} = 12e^{2x}$

$\therefore \frac{d^2y}{dx^2} - \frac{2dy}{dx} + y$

$12e^{2x} - 2(6e^{2x}) + 3e^{2x}$ 3

$= 3e^{2x}$

b)

x	$f(x)$	$wf(x)$	$f(x)\times wf(x)$
0	2	1	2
1	3	4	12
2	12	2	24
3	35	4	140
4	80	1	80
			258

$\therefore \int_0^4 f(x)\,dx \doteqdot \frac{h}{3} \times f(x).wf(x)$

$\doteqdot \frac{1}{3} \times 258$ 3

$\doteqdot 86$

c) Point of intersection 2

i) $-x^2 = 3x$ $\quad \therefore x = 0$ and $x = 3$

$3x + x^2 = 0$ $\quad P_1(0,0)$ and $P_2(-3,-9)$

$x(3+x) = 0$ $\quad \therefore A(-3,-9)$

c) ii) Area $= \left| \int_{-3}^{0} -x^2 - 3x\,dx \right|$

$= \left| \left[\frac{-x^3}{3} - \frac{3x^2}{2} \right]_{-3}^{0} \right|$

$= 0 - \left[9 - \frac{27}{2} \right]$ 3

Area $= 4\frac{1}{2}$ units^2

d) Evaluate $\int_a^c f(x)\,dx = 2A - 4 - (A+4)$

$= 2A - 4 - A - 4$

$= A - 8$ 2

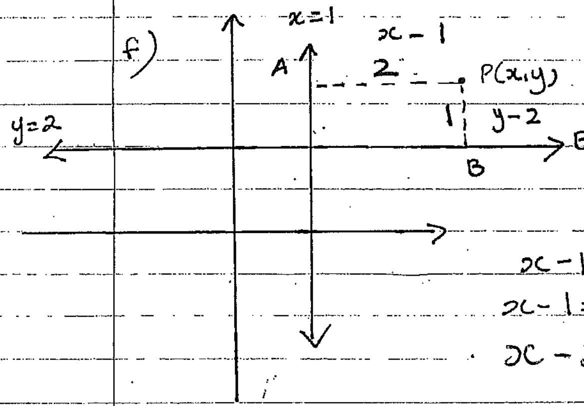
e) $x^2 - 2x + y^2 - 4y - 11 = 0$

$x^2 - 2x + 1 + y^2 + 4y + 4 = 11 + 5$ 3

$(x-1)^2 + (y+2)^2 = 16$

$\therefore$ Centre $(1,\bar{2})$ and radius = 4 units.

f)



$PA = 2PB$

$\sqrt{(x-1)^2} = 2\sqrt{(y-2)^2}$ 4

$(x-1) = \pm 2(y-2)$

$x - 1 = 2(y-2)$ or $(x-1) = -2(y-$

$x - 1 = 2y - 4$ $\quad x - 1 = -2y + 4$

$x - 2y + 3 = 0$ $\quad x + 2y - 5 = 0$

$\therefore$ Solution: $x - 2y + 3 = 0$

$x + 2y - 5 = 0$

3a) i)

$y = 3x$

$V = \pi\int_0^3 9x^2\,dx$ (1)

ii)

x	$f(x)$	y^2	w	wy^2
0	0	0	1	0
1	3	9	2	18
2	6	36	2	72
3	9	81	1	81

$\Sigma\ 171$

$V = \frac{1}{2} \times 171\pi$

$= 85.5\pi\ u^3$

$= 268.61$ (2)

iii) $V = \pi\int_0^3 9x^2\,dx$

$= \pi\left[\frac{9x^3}{3}\right]_0^3$

$= \pi\left[3x^3\right]_0^3$

$= 81\pi\ u^3$ (3)

$= 254.47\ u^3$

% error $= \frac{14.14}{254.47} \times 100\% = 5.56\%$

b) i) $y = 4xe^{-2x}$

$y = 0$

$4x = 0$

$x = 0$ (1)

ii) $y' = 4e^{-2x} + -2x\,4xe^{-2x}$

$= 4e^{-2x} - 8xe^{-2x}$ (2)

iii) $4e^{-2x}(1 - 2x) = 0$

$1 - 2x = 0$

$x = \frac{1}{2}$

$x = \frac{1}{2} \Rightarrow y = 2e^{-1}$

$= \frac{2}{e}$ (2)

$\left(\frac{1}{2}, \frac{2}{e}\right)$

$y'' = -8e^{-2x} - (8e^{-2x} + 16xe^{-2x})$

$= -16e^{-2x} + 16xe^{-2x}$

at $x = \frac{1}{2}$

$y'' = -\frac{16}{e} + \frac{8}{e}$

$= -\frac{8}{e} < 0$

max

iv) $y'' = 0$

$-16e^{-2x}(1 - x) = 0$

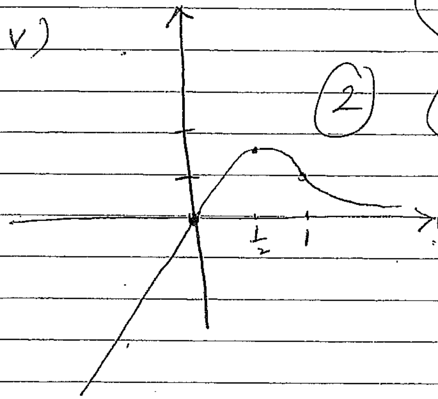
$x = 1$

$y = \frac{4}{e^2}$ (3)

x	0	1	2
y'	$-$	0	$+$

$\therefore \left(1, \frac{4}{e^2}\right)$ pt of inflexion

v)



(2)

c) $y' = 3e^{3x}$

at $x = 2$

$m_1 = 3e^6$

$m_2 = -\frac{1}{3e^6}$

$y - e^6 = -\frac{1}{3e^6}(x - 2)$

$3e^6 y - 3e^{12} = -x + 2$

$x + 3e^6 y - 3e^{12} - 2 = 0$ (2)

$y = 0$

$x = 3e^{12} + 2$

(2)

$T(3e^{12} + 2, 0)$

Question 4

a) i) $y^2 - 4y - 4x - 8 = 0$

$$y^2 - 4y = 4x + 8$$

$$y^2 - 4y + 4 = 4x + 12$$

$$(y-2)^2 = 4(x+3) \quad ① \qquad \Rightarrow \quad 2.$$

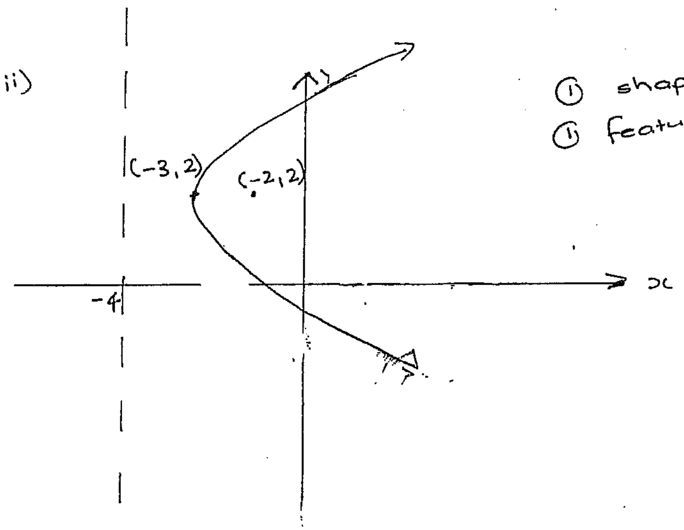
$①$

vertex $(-3, 2)$

$4a = 4$

$a = 1$

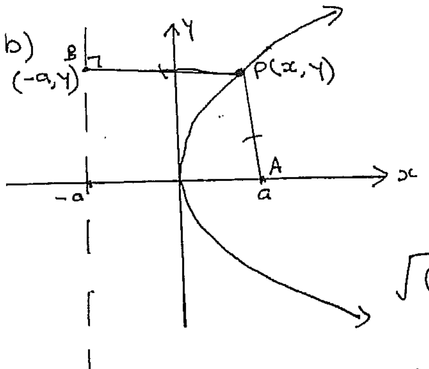
ii)



① shape $\Rightarrow$ 2

① features

b)



$$PA = \sqrt{(x-a)^2 + (y-0)^2}$$

$$= \sqrt{(x-a)^2 + y^2} \quad ①$$

$$PB = \sqrt{(x+a)^2} \quad ①$$

$\Rightarrow 4$

As $PA = PB$

$$\sqrt{(x-a)^2 + y^2} = \sqrt{(x+a)^2} \quad ①$$

$$(x-a)^2 + y^2 = (x+a)^2$$

$$\cancel{x^2} - 2ax + \cancel{a^2} + y^2 = \cancel{x^2} + 2ax + \cancel{a^2}$$

$$y^2 = 4ax \quad ①$$

c) i)

$$V = \pi \int_0^9 \left(\sqrt{9-x}\right)^2 dx$$

$①$

$$= \pi \int_0^9 (9-x)\, dx$$

$① \Rightarrow 3$

$$= \pi \left[9x - \frac{x^2}{2} \right]_0^9$$

$$= \pi \left\{ \left(9 \times 9 - \frac{9^2}{2}\right) - \left(9 \times 0 - \frac{0^2}{2}\right) \right\}$$

$①$

$$= \frac{81\pi}{2} \text{ cm}^3$$

ii) Vol. in each glass $= \frac{2}{3} \times \frac{81\pi}{2}$

$$= 27\pi \text{ cm}^3$$

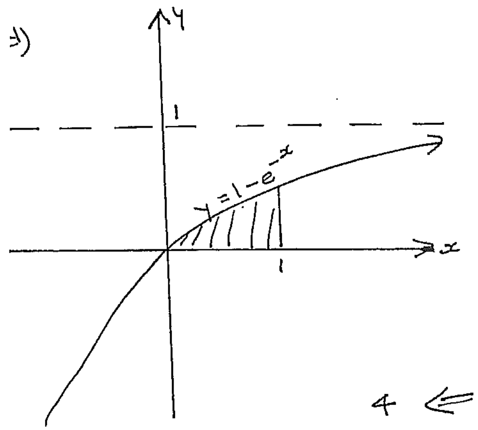
$$\doteqdot 84 \cdot 82 \text{ cm}^3$$

Number of glasses $= \frac{1000}{27\pi}$

$$= 11.789 \ldots \quad ①$$

$\therefore$ 11 glasses can be filled.

d)



$$V = \pi \int_0^1 \left(1 - e^{-x}\right)^2 dx \quad ①$$

$$= \pi \int_0^1 \left(1 - 2e^{-x} + e^{-2x}\right) dx \quad ①$$

$$= \pi \left[x + 2e^{-x} - \frac{1}{2} e^{-2x} \right]_0^1 \quad ①$$

$$= \pi \left\{ \left(1 + 2e^{-1} - \frac{1}{2}e^{-2}\right) - \left(0 + 2e^{0} - \frac{1}{2}e^{0}\right) \right\}$$

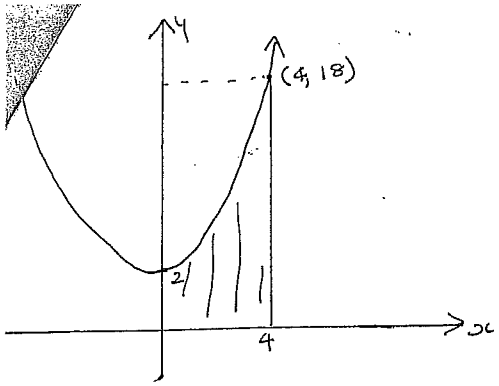
$4 \Leftarrow$

$$= \pi \left\{ \left(1 + \frac{2}{e} - \frac{1}{2e^2}\right) - \left(\frac{3}{2}\right) \right\}$$

$$= \pi \left(\frac{2}{e} - \frac{1}{2e^2} - \frac{1}{2} \right)$$

$$= \frac{\pi}{2} \left(\frac{4}{e} - \frac{1}{e^2} - 1 \right) u^3 \quad ①$$

$$= \pi \, (4e - 1 - e^2)$$



Vol. around y axis:

$$V = \pi \int_2^{18} (y-2)\,dy$$

$$= \pi \left[\frac{y^2}{2} - 2y\right]_2^{18}$$

$$= \pi \left\{\left(\frac{18^2}{2} - 2\times18\right) - \left(\frac{2^2}{2} - 2\times2\right)\right\}$$

$$= 128\pi \ u^3 \quad 2$$

Vol. of cylinder $= \pi \times 4^2 \times 18$

$= 288\pi \ u^3$ 1 $\Rightarrow$ 4.

$\therefore$ Reqd. volume $= 288\pi - 128\pi$

$= 160\pi \ u^3$ 1