Sydney Girls High School
2012
Trial Higher School Certificate
Examination

Mathematics Extension 1

General Instructions

- Reading Time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- Show all necessary working in Questions 11 – 14

This is a trial paper ONLY. It does not necessarily reflect the format or the contents of the 2012 HSC Examination Paper in this subject.

Total marks – 70

SECTION 1 – Pages 2 - 5
10 marks

- Attempt questions 1 – 10
- Allow about 15 minutes for this section

SECTION II – Pages 6 - 9
60 marks

- Attempt questions 11 – 14
- Allow about 1 hours 45 minutes for this section

Name: ______________________

Teacher: ______________________

Section I - Total Marks 10

Attempt Questions 1 – 10
Allow about 15 minutes for this section.

(1) Find $\lim_{x\to 0}\frac{\sin 3x}{4x}$

(a) $\frac{4}{3}$

(b) 1

(c) $\frac{3}{4}$

(d) 0

(2) $\frac{d}{dx}[\sin(\log x)] =$

(a) $\cos(\log x)$

(b) $\frac{\cos(\log x)}{x}$

(c) $\frac{\sin(\log x)}{x}$

(d) $-\cos(\log x)$

(3) We can express $\sin x$ and $\cos x$ in terms of $\tan\frac{x}{2}$, for all values of x except

(a) $x = 2\pi, 6\pi, 8\pi, \ldots$

(b) $x = \pi, 3\pi, 5\pi, \ldots$

(c) $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

(d) $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$

(4) Which of the following is an expression for $\int \cos^2 8x dx$

(a) $\frac{x}{2} - \frac{\sin 8x}{32} + c$

(b) $\frac{x}{2} + \frac{\sin 8x}{32} + c$

(c) $\frac{x}{2} - \frac{\sin 16x}{32} + c$

(d) $\frac{x}{2} + \frac{\sin 16x}{32} + c$

(5) Which of the following is the correct expression for $\int \frac{dx}{\sqrt{36-4x^2}}$

(a) $\frac{1}{2}\sin^{-1}\frac{x}{6}$

(b) $\frac{1}{2}\sin^{-1}\frac{x}{3}$

(c) $\frac{1}{4}\sin^{-1}\frac{x}{6}$

(d) $\frac{1}{6}\sin^{-1}\frac{x}{3}$

(6) The velocity, v metres per second, of a particle moving in simple harmonic motion along the x-axis is given by the equation $v^2 = 64 - 16x^2$
What is the period, in seconds of the motion of the particle?

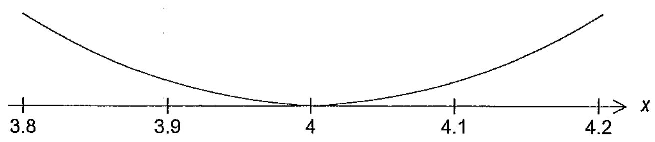
(a) $\frac{\pi}{8}$

(b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$

(d) π

(7) Part of the graph of $y = P(x)$, where $P(x)$ is a polynomial of degree three, is shown below.



Which of the following could be the polynomial $P(x)$?

(a) $(x-4)^3$

(b) $(x-5)(x+4)^2$

(c) $(x-1)(x-4)^2$

(d) $(x-1)(x+2)(x-4)$

(8) The radius of a sphere is increasing at a rate of 5 centimetres per minute.

What is the rate of increase of the surface area of the sphere, in cubic centimetres per minute, when the radius is 4 centimetres?

(a) 32π

(b) 64π

(c) 100π

(d) 160π

(9) Which of the following represents the inverse function of $f(x) = \frac{5}{2x-6} - 2$

(a) $f^{-1}(x) = \frac{5}{2x+4} + 3$

(b) $f^{-1}(x) = \frac{5}{2x+4} - 3$

(c) $f^{-1}(x) = 3 - \frac{5}{2x+4}$

(d) $f^{-1}(x) = \frac{5}{x+2} + 6$

(10) How many solutions does the equation $\sin 2\theta = \cos\theta$ have in the domain $0 \le \theta \le 2\pi$?

(a) 4

(b) 3

(c) 2

(d) 1

END OF SECTION I

Section II - Total Marks 60

Attempt Questions 11 – 14
Allow about 1 hour 45 minutes for this section

Answer all questions, starting each question on a new sheet of paper

Question 11 (15 marks) **Marks**

(a) Find the acute angle between the lines (to the nearest degree) 2
$3x+2y-6=0$ and $2x-y+8=0$

(b) A curve has parametric equations $x=\frac{t}{3}, y=2t^2$. 2
Find the Cartesian equation of this curve.

(c) Find $\int 2x\sqrt{x-5}\ dx$ using the substitution $u = x-5$ 3

(d) If α, β and γ are the roots of the polynomial $5x^3-2x-4=0$, find 2
the value of $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$.

(e) Find the exact value of $\tan^{-1}(-\sqrt{3})$. 1

(f) Find the coordinates of the point P that divides the interval AB externally 2
in the ratio 3 : 2, where the coordinates of A and B are respectively $(-2, 4)$
and $(3, -6)$.

(g) Solve $\frac{x}{2-x} \ge 2$. 3

Question 12 (15 marks) - Start a new page **Marks**

(a) Prove by induction, that $5^n > 20n-1$ for $n \geq 3$, where n is an integer. 3

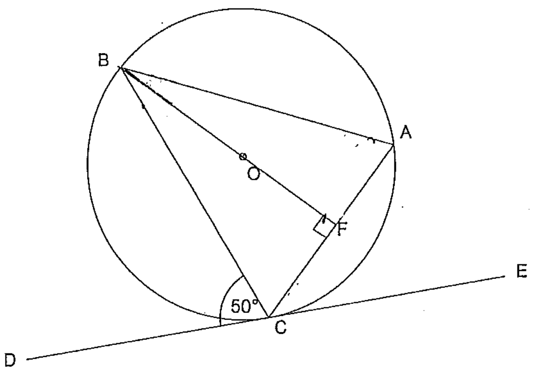
(b) The probability that it rains on any particular day in London is $\frac{2}{3}$.

(i) What is the probability that it does not rain for a whole week in London? 2

(ii) What is the probability that it will rain on only two days during a whole week in London and that these two days are consecutive? 2

(c) Evaluate $\int_0^{\frac{\pi}{2}} \cos x \sin^2 x \, dx$ 2

(d) The line DE is tangent to the circle at C. If $DCB = 50^0$, find the size of $\angle ACE$ giving full reasons. 2



(e) Sketch the graph of $y = sin^{-1}(x-2)$ 2

(f) The function $f(x) = \sin x - \frac{x}{2}$ has a zero near $x = 2$ 2

Taking $x = 2$ as a first approximation, use one application of Newton's method to find a second approximation to the zero. Give your answer correct to two decimal places.

Question 13 (15 marks) Start a new page **Marks**

(a) Find the exact value of $\int_0^2 \frac{dx}{4x^2+20}$ 2

(b) An iron is cooling in a room of constant temperature 20^0C. At time t minutes its temperature T decreases according to the equation $\frac{dT}{dt} = -k(T-20)$ where k is a positive constant.

The initial temperature of the iron is 100^0C and it cools to 70^0C after 15 minutes.

(i) Verify that $T = 20 + Ae^{-kt}$ is a solution of this equation, where A is a constant. 1

(ii) Find the values of A and k. 2

(iii) How long will it take for the temperature of the iron to cool to to 25^0C? 2

(c) Calculate the exact volume generated by the solid formed when $y = cos^{-1}x$ is rotated about the *y-axis* between $y = 0$ and $y = \pi$. 3

(d) $P(x) = x^4 + 5x^3 + 4x^2 - 8x - 8$

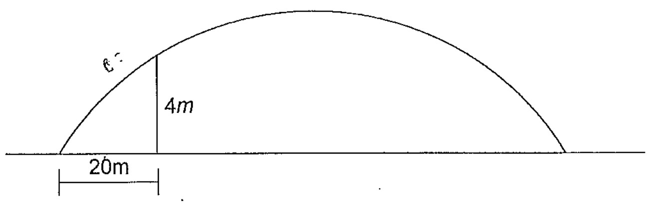
i. Show that $(x+1)(x+2)$ is a factor of $P(x)$ 1

ii. Find $Q(x)$ if $P(x) = (x+1)(x+2)Q(x)$ 1

(e) Solve the equation $\tan\theta = \sin 2\theta, 0 \leq \theta \leq 2\pi$ 3

Question 14 (15 marks) Start a new page **Marks**

(a)



A projectile is fired with initial velocity Vms^{-1} at an angle of θ from a point O on horizontal ground. After 5 seconds it just passes over a 4m high wall that is 20 metres from the point of projection. Assume the acceleration due to gravity is $10ms^{-2}$
Assume the equations of displacement are $x = Vt\cos\theta$ and $y = Vt\sin\theta - 5t^2$.

(i) Find V and θ 2

(ii) Find the time taken for the projectile to attain its maximum height. 1

(iii) Find the range of the projectile. 1

(b) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

(i) Show that the equation of the chord PQ is $(p+q)x - 2y - 2apq = 0$ 2

(ii) Show that the gradient of the tangent at P is p. 2

(iii) Prove that if the tangent at P is parallel to the normal at Q then PQ passes through the focus S 2

(c) A particle moves in a straight line so that its acceleration is given by $a = x-2$, where x is its displacement from the origin.
Initially, the particle is at the origin and has velocity $v = 2$

(i) Find the initial acceleration. 1

(ii) Show that $v^2 = (x-2)^2$ 2

(iii) Find x as a function of t. 2

--End of Exam--

EX1, TRIAL 2012

(1) C $\lim_{x\to 0} \frac{\sin 3x}{3x} \times \frac{3}{4}$

(2) B $\cos(\log x) \times \frac{1}{x}$

(3) B $\frac{x}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

(4) D $\cos 16x = \cos^2 8x - \sin^2 8x$

$= \cos^2 8x - (1 - \cos^2 8x)$

$= 2\cos^2 8x - 1$

$\frac{\cos 16x + 1}{2} = \cos^2 8x$

(5) B $\frac{1}{2}\int \frac{dx}{\sqrt{9-x^2}}$

(6) C $\frac{1}{2}v^2 = 32 - 8x^2$

$\ddot{x} = -16x$

$n^2 = 16$

$n = 4$

$T = \frac{2\pi}{4}$

(7) C $x = 4$ is a double root

(8) D $S = 4\pi r^2$

$\frac{dS}{dr} = 8\pi r$

$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$

$= 8 \times \pi \times 4 \times 5$

(9) A $x = \frac{5}{2y-6} - 2$

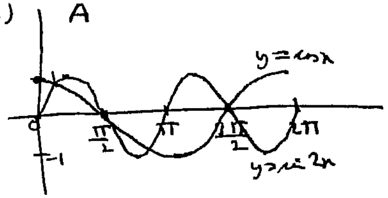
$x + 2 = \frac{5}{2y-6}$

$2y - 6 = \frac{5}{x+2}$

$2y = \frac{5}{x+2} + 6$

$y = \frac{5}{2x+4} + 3$

(10) A



Ext 1 2012

11

a) $\tan\theta = \left|\frac{-\frac{3}{2} - 2}{1 + (-\frac{3}{2} \times 2)}\right|$

$\theta = 60^\circ$ (2)

b) $t = 3x$

$y = 2(3x)^2$

$= 2(9x^2)$

$y = 18x^2$ (2)

c) $u = x - 5$

$\frac{du}{dx} = 1$

$du = dx$

$\int 2(u+5)\sqrt{u}\, du$

$= \int (2u + 10)\, u^{\frac{1}{2}}\, du$

$= \int 2u^{\frac{3}{2}} + 10u^{\frac{1}{2}}\, du$ (3)

$= \frac{2u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{10u^{\frac{3}{2}}}{\frac{3}{2}} + C$

$= \frac{4}{5}\sqrt{(x-5)^5} + \frac{20}{3}\sqrt{(x-5)^3} + C$

d) $\dfrac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$ 2

$= \dfrac{-1}{2}$

e) $\tan^{-1}(-\sqrt{3}) = \dfrac{-\pi}{3}$ 1

f) $P = \dfrac{-3(3) + 2(-2)}{-3+2}, \dfrac{-3(-6)+2(4)}{-3+2}$

2

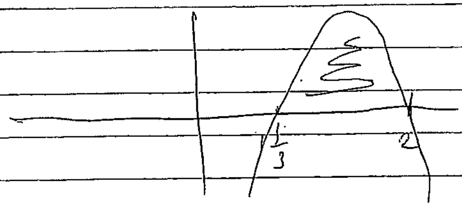
$P = (13, -26)$

g) $\dfrac{x}{2-x} - 2 \geq 0$ 3

$\dfrac{2-x-2(2-x)}{2-x} \geq 0$

$(3x-4)(2-x) \geq 0$

$1\frac{1}{3} \leq x \leq 2$



Question 12.

a) Prove $5^n > 20n - 1$

I, Prove true when $n = 3$

$LHS = 5^3$ $\quad$ $RHS = 20 \times 3 - 1$

$= 125$ $\quad$ $= 59$

$LHS > RHS$

$\therefore$ true when $n = 3$.

II, Assume true for $n = k$.

$5^k > 20k - 1$

III, Prove true for $n = k+1$.

RTP $5^{k+1} > 20(k+1) - 1$.

i.e. Prove $LHS - RHS > 0$

$LHS - RHS = 5^{k+1} - 20(k+1) + 1$

$= 5^k . 5 - 20k - 20 + 1$

$> 5(20k-1) - 20k - 19$

$= 100k - 5 - 20k - 19$

$= 80k - 24$

> 0 as when $k \geq 3$, $80k - 24 \geq 216$

IV, Blah.

b) i) P(docs not rain for a week) $= \left(\frac{1}{3}\right)^7$

$= \dfrac{1}{2187}$

ii) $P(SM) + P(MT) + P(TW) + P(WT) + P(TF) + P(FS)$

$P(SM) + P(MT) + P(TW) + P(WT) + P(TF) + P(FS)$

$= \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{3}\right)^5 \times 6$

$= \dfrac{8}{729}$

c) $u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x \, dx$

$= 1$

when $x = 0$; $u = \sin 0$

$= 0$

$$I = \int_0^1 u^2 \, du$$

$$= \left[\frac{u^3}{3}\right]_0^1$$

$$= \frac{1^3}{3} - \frac{0^3}{3}$$

$$= \frac{1}{3}$$

d) $\angle BAC = \angle BCD$ ($\angle$ in alt. segment)

$= 50^\circ$

In ΔBAF, $\angle ABF = 40^\circ$ ($\angle$ sum Δ)

Join OC

$\angle OCD = 90^\circ$ (tangent $\perp$ radius)

$\therefore \angle OCB = 40^\circ$ (comp. $\angle$s)

As $OC = OB$ (radii)

ΔOCB is isosceles

$\therefore \angle OBC = 40^\circ$ (base $\angle$s of isos Δ)

$\angle ABC = \angle ABF + \angle FBC$ (adj. $\angle$s)

$= 40^\circ + 40^\circ$

$= 80^\circ$

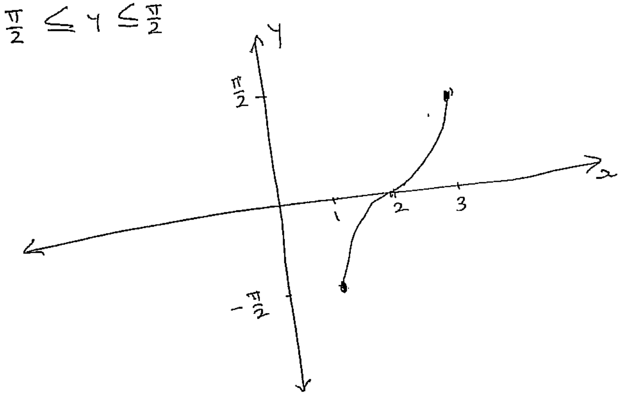
$\angle ACE = \angle ABC$ ($\angle$ in alt. segment)

$= 50^\circ$

D: $-1 \leq x - 2 \leq 1$

$1 \leq x \leq 3$

R: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



f) $f(x) = \sin x - \frac{x}{2}$

$f'(x) = \cos x - \frac{1}{2}$

$f(2) = \sin(2) - \frac{2}{2}$

$= \sin(2) - 1$

$= -0.00907\ldots$

$f'(2) = \cos(2) - \frac{1}{2}$

$= -0.91615\ldots$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2 - \frac{f(2)}{f'(2)}$$

$$= 2 - \frac{-0.00907\ldots}{-0.91615}$$

$= 1.900995594\ldots$

$= 1.90$ (2 dec. pl)

Question 13

(a) $$\int_0^2 \frac{dx}{4x^2+20} = \frac{1}{4}\int_0^2 \frac{dx}{x^2+5}$$

$$= \frac{1}{4\sqrt{5}}\left[\tan^{-1}\frac{x}{\sqrt{5}}\right]_0^2$$

$$= \frac{\tan^{-1}\left(\frac{2}{\sqrt{5}}\right)}{4\sqrt{5}}$$

(b) (i) $T = 20 + Ae^{-kt}$

$$\text{LHS} = \frac{dT}{dt}$$

$$= -kAe^{-kt}$$

$$= -k\left(20 + Ae^{-kt} - 20\right)$$

$$= -k(T-20)$$

$\therefore$ LHS $=$ RHS $\quad\therefore$ $T = 20 + Ae^{-kt}$ is a soln. of the equation

(ii) when $t=0$, $T=100$ $\quad$ when $t=15$, $T=70$

$$100 = 20 + A \Rightarrow A = 80$$

$$70 = 20 + 80e^{-15k}$$

$$e^{-15k} = \frac{5}{8} \Rightarrow k = -\frac{1}{15}\ln\left(\frac{5}{8}\right) \doteqdot 0.031$$

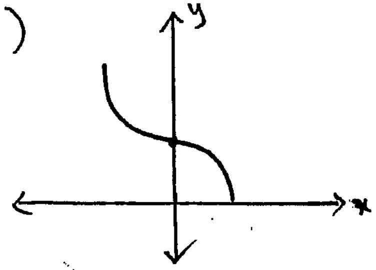
(iii) $$25 = 20 + 80e^{-0.031t}$$

$$e^{-0.031t} = \frac{5}{80} \Rightarrow t = \frac{-1}{0.031}\ln\left(\frac{5}{80}\right)$$

$\therefore t \doteqdot 88.5$ minutes

Question 13

(c)



$$V = \pi\int_0^{\pi} x^2\,dy$$

$$= \pi\int_0^{\pi} \cos^2 y\,dy$$

$$= \frac{\pi}{2}\int_0^{\pi} (\cos 2y + 1)\,dy$$

$$= \frac{\pi}{2}\left[\frac{\sin 2y}{2} + y\right]_0^{\pi}$$

$$= \frac{\pi}{2}\left[0 + \pi - (0+0)\right]$$

$$\therefore V = \frac{\pi^2}{2} \text{ units}^3$$

(d) (i) $P(x) = x^4 + 5x^3 + 4x^2 - 8x - 8$

$$P(-1) = 1 - 5 + 4 + 8 - 8$$
$$= 0 \quad \therefore (x+1) \text{ is a factor}$$

$$P(-2) = 16 - 40 + 16 + 16 - 8$$
$$= 0 \quad \therefore (x+2) \text{ is a factor}$$

Hence $(x+1)(x+2)$ is a factor

(ii) $$(x^2 + 3x + 2)\,Q(x) = x^4 + 5x^3 + 4x^2 - 8x - 8$$

By inspection $Q(x) = x^2 + 2x - 4$

Question 13

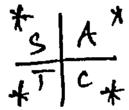
(e) $\tan\theta = \sin 2\theta$

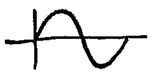
$$\frac{\sin\theta}{\cos\theta} = 2\sin\theta\cos\theta$$

$$2\sin\theta\cos^2\theta - \sin\theta = 0$$

$$\sin\theta(2\cos^2\theta - 1) = 0$$

$\sin\theta = 0$ or $\cos\theta = \pm\frac{1}{\sqrt{2}}$





$\left(\text{acute } \angle = \frac{\pi}{4}\right)$

$$\therefore\ \theta = 0, \pi, 2\pi, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Question 14

a) $x = Vt\cos\theta$ $\qquad y = Vt\sin\theta - 5t^2$

i) When $x = 20$,
$y = 4$
$t = 5$

$20 = 5V\cos\theta$
$4 = V\cos\theta$
$V = \frac{4}{\cos\theta}$ (1)

$4 = 5V\sin\theta - 125$
$\frac{129}{5} = V\sin\theta$ (2)

Sub (1) into (2)

$\frac{129}{20} = \tan\theta$

$\theta = \tan^{-1}\left(\frac{129}{20}\right)$
$= 81°11'$

$V = \frac{4}{\cos 81°11'}$

$= 26.1\ ms^{-1}$

ii) Max height when $y' = 0$
$y' = V\sin\theta - 10t$
$V\sin\theta = 10t$
$t = \frac{V\sin\theta}{10}$

$= 2.58\ s.$

iii) Time of flight $= 2 \times 2.58$
$= 5.16$

Range $= 26.1 \times 5.16 \times \cos 81°11'$
$= 20.64$

b) $P(2ap, ap^2)$ $\qquad Q(2aq, aq^2)$

i) $M_{PQ} = \frac{ap^2 - aq^2}{2ap - 2aq}$

$= \frac{a(p+q)(p-q)}{2a(p-q)}$

$= \frac{p+q}{2}$

Equation of PQ:
$y - ap^2 = \frac{p+q}{2}(x - 2ap)$

$2y - 2ap^2 = (p+q)x - 2ap^2 - 2apq$

$(p+q)x - 2y - 2apq = 0$

ii) $x^2 = 4ay$
$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a}$
$= \frac{x}{2a}$

when $x = 2ap$
$\frac{dy}{dx} = \frac{2ap}{2a}$
$= p.$

iii) If tangent at P is parallel to normal at Q then gradient of normal at Q is p and gradient of tangent at Q is $\frac{-1}{p}$.

$\therefore q = \frac{-1}{p}$

Substituting into PQ:
$(p+q)x - 2y + 2a = 0$
when $x = 0$
$-2y + 2a = 0$
$y = a$

$\therefore$ PQ passes through $(0, a)$

c)

i) $a = x - 2$

when $t = 0$, $x = 0$

$\therefore a = -2$

ii) $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x - 2$

$\frac{1}{2}v^2 = \int x - 2 \, dx$

$v^2 = 2\int x - 2 \, dx$

$= \frac{\cancel{2}(x-2)^2}{\cancel{2}} + C$

When $x = 0$, $v = 2$

$4 = 4 + C$

$C = 0$

$\therefore v^2 = (x-2)^2$

iii) when $x = 0$ $v = 2$

$\therefore v = -(x-2)$

$v = 2 - x$

$\frac{dx}{dt} = 2 - x$

$\frac{1}{2-x} dx = dt$

$t = -\ln(2-x) + C$

when $t = 0$, $x = 0$

$0 = -\ln 2 + C$

$C = \ln 2$

$t = -\ln(2-x) + \ln 2$

$\ln 2 - t = \ln(2-x)$

$x = 2 - e^{\ln 2 - t}$

$= 2 - 2e^{-t}$