Sydney Girls High School

November 2015

MATHEMATICS EXTENSION 2

YEAR 12
ASSESSMENT TASK 1 for HSC 2016

Time Allowed: 60 minutes + 5 minutes reading time

Total: 48 marks

Topics: Circular Motion, Curve Sketching

There are FOUR (4) Questions which are of equal value
Attempt all questions
Show all necessary working. Marks may be deducted for badly arranged work or incomplete working
Start each question on a new page.
Write on one side of the paper only.
Diagrams are NOT to scale.
Board-approved calculators may be used
Use $g = 10\ ms^{-2}$

Student Name:____________________ **Teacher Name:**_______________

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Question 1 (12 marks) | **Marks**

(a) Sketch the following without using calculus and showing all important features on separate number planes.

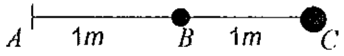
i) $y = |1 - x^3|$ — 2

ii) $4x^2 - y^2 = 8$ — 2

iii) $y = x^3(16 - x^3)$ — 2

(b) A string of length 120cm is attached to a fixed point on a smooth table. An object of mass 3 kg is attached at the end of the string. Find the tension in the string if the object is rotated with uniform circular motion at a speed of 5 metres per second. — 2

(c) A 2 metre piece of string ABC has a mass of 2 kg at point B and 3 kg at point C.



The masses are placed on a smooth table with the string attached to the table at A. The system is rotated about A and the string breaks when it reaches a speed of rotation of 5 radians per second.

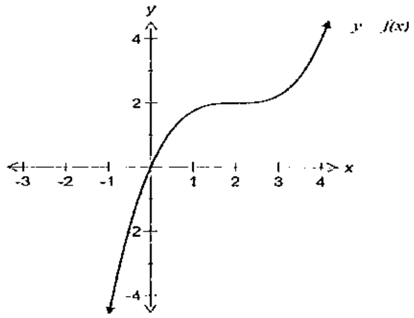
i) Find the breaking strain of the string. — 2

ii) If the masses at B and C are now swapped, find the new maximum angular speed of rotation. — 2

Question 2 (Start a new page) (12 marks) | **Marks**

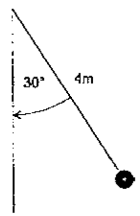
(a) Graph the curve $\dfrac{(x-1)^2}{16} + \dfrac{y^2}{9} = 1$. — 2

(b) Copy the graph of $y = f(x)$ and on the same diagram, clearly sketch the inverse function $y = f^{-1}(x)$. — 2



(c) By considering the graphs of $y = \sin x$ (for $-2\pi \le x \le 2\pi$) and $y = 3^x$, sketch the graph of $y = \sin x + 3^x$. — 2

(d) A heavy mass at the end of a light rod is rotated in a circle, with the motion resembling a conical pendulum, making an angle of 30^0 with the vertical. The length of the light rod is 4 metres.



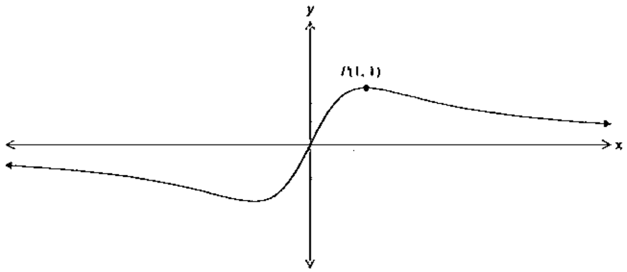
i) Find the time to complete one revolution, correct to 1 decimal place. — 3

ii) Find the velocity of the mass. — 1

iii) If the angular speed is trebled, find the new angle with the vertical, to the nearest degree. — 2

Question 3 (Start a new page) (12 marks) **Marks**

(a) The graph of the odd function $y = f(x)$ is shown below.



Sketch the following, showing all important features on separate number planes.

i) $y = f(|x|)$ 1

ii) $y = [f(x)]^2$ 1

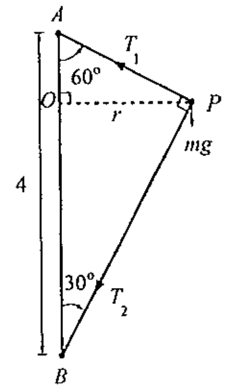
iii) $y = \dfrac{1}{f(x)}$ 2

iv) $y = 2^{f(x)}$ 2

Question 3 continues on the next page

Question 3 (continued) **Marks**

(b) A particle P, of mass 4 kg is connected by light rods attached to the fixed points A and B as shown in the diagram below. The point B is vertically below A at a distance of 4 metres and $\angle PAB = 60^\circ$ and $\angle PBA = 30^\circ$.



The forces acting on the particle at P are T_1 (the tension in the rod AP), T_2 (the tension in the rod BP) and mg (gravity).

i) Show that the radius $OP = \sqrt{3}$ metres. 2

ii) If the particle rotates at 30 rev/min about O, show that the tension in rods AP and BP are respectively: 4

$$T_1 = 2(g + 3\pi^2)$$

$$\text{and } T_2 = 2\sqrt{3}(\pi^2 - g)$$

Question 4 (Start a new page) (12 marks) **Marks**

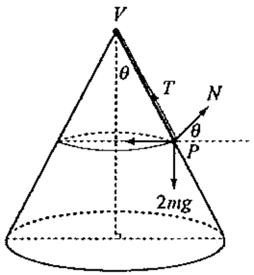
(a) For the function $f(x)=\dfrac{x^2+2x+4}{x^2-x-6}$

i) Find the equations of all asymptotes of $y=f(x)$. 2

ii) Sketch $y=f(x)$ clearly showing the asymptotes and any intercepts. 2

(b) Sketch $y=x\sqrt{4-x^2}$ 2

(c) A circular cone of semi-vertical angle θ is fixed with its vertex upwards. A particle P of mass $2m$ kg is attached to the vertex V by a light inextensible string of length $2a$ metres.
The particle P rotates with uniform angular velocity ω in a horizontal circle on the outside surface of the cone and in contact with it.



i) Show that the tension in the string is given by: $T=2m(g\cos\theta+2a\omega^2\sin^2\theta)$. 4

ii) Derive an expression for the normal force N acting on P. 1

iii) Show that for the particle to remain in steady motion on the surface of the cone, then: 1

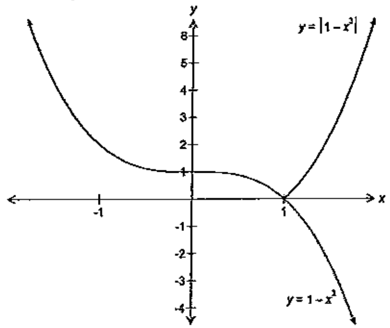
$$\omega<\left(\frac{g}{2a\cos\theta}\right)^{\frac{1}{2}}$$

End of Paper

Mathematics Extension 2 HSC Task 1 Nov 2015 Solutions

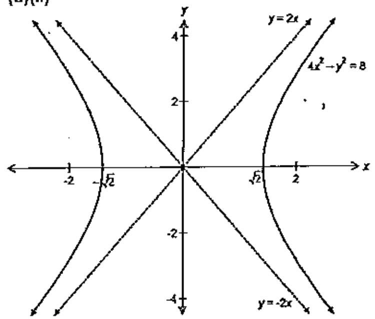
QUESTION 1

(a)(i)



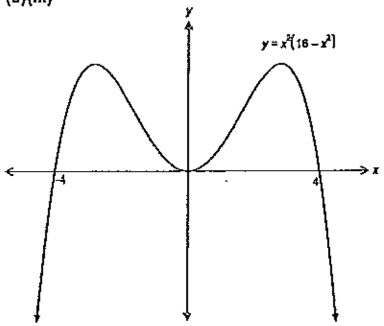
**The concavity of this graph is a key element. Some students reflected the graph of $y = 1 - x^3$ but did not identify the correct shaping when $x > 1$.
Some students did not identify the inflexion point at $(0,1)$ or the sharp point at $(1,0)$.

(a)(ii)



**Many students produced poor quality sketches and did not show the curve approaching the two asymptotes.
Some students did not correctly identify the equations of the asymptotes.

(a)(iii)



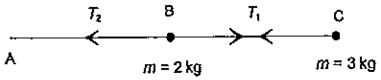
**A range of students chose to approach this by sketching the product of $y = x^2$ and $y = 16 - x^2$. Recognising the function as a polynomial is easier.

(b) $m = 3\text{ kg}$; $r = 1.2\text{ m}$; $v = 5\text{ m/s}$

$$T = \frac{mv^2}{r} = \frac{3\times 5^2}{1.2} \qquad \therefore T = 62.5\text{ N}$$

**Generally well done though some students forgot to convert the radius into the correct units (m) or incorrectly identified 5 as the *angular* speed.

(c)(i) $\omega = 5\text{ rad/s}$; $T = mr\omega^2$



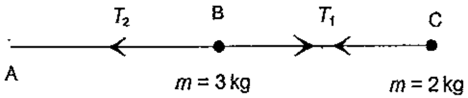
For 3 kg mass : $T_1 = 3\times 2\times 5^2 = 150\text{ N}$

For 2 kg mass : $T_2 - T_1 = 2\times 1\times 5^2 = 50\text{ N}$

Breaking strain : $T_2 = 50 + T_1 = 200\text{ N}$

**Some solutions did not demonstrate a clear understanding of the relationship between the forces including the tension forces, including that the greatest tension occurs closest to the centre of the system. Some students incorrectly added T_1 and T_2.

(c)(ii) Maximum angular speed $\omega = ?$ when $T_2 = 200\text{ N}$; $T = mr\omega^2$



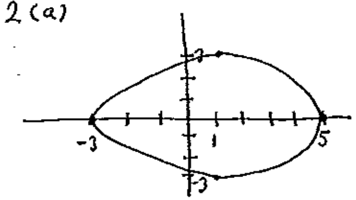
For 2 kg mass : $T_1 = 2\times 2\times \omega^2 = 4\omega^2$

For 3 kg mass : $T_2 - T_1 = 3\times 1\times \omega^2 = 3\omega^2 \quad \therefore T_2 = 3\omega^2 + T_1 = 7\omega^2$

$7\omega^2 = 200 \qquad \therefore \omega = \sqrt{\frac{200}{7}} \approx 5.35\text{ rad/s}$

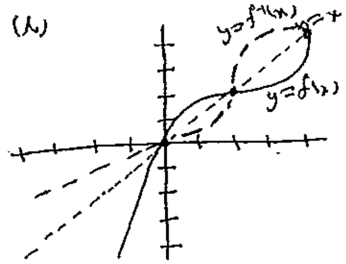
**Using the answer from (i), students needed to calculate the value of the angular speed ω.

2 (a)



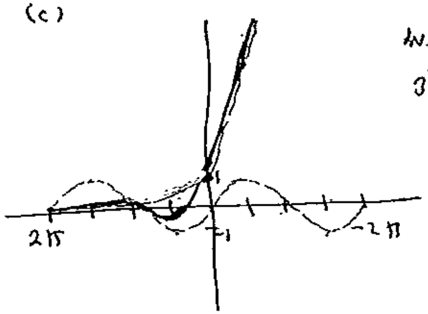
the y intercepts are not ±3

(b)



f(x) and f⁻¹(x) intersect in 3 places on y=x

(c)



when x > 0
3^x dominates

(d)(i) $\cos 30° = \frac{h}{4}$

$\frac{\sqrt{3}}{2} = \frac{h}{4}$

$h = 2\sqrt{3}$

$2\sqrt{3} = \frac{g}{w^2}$

$w^2 = \frac{10}{2\sqrt{3}}$

$w = \left(\frac{5}{\sqrt{3}}\right)^{\frac{1}{2}}$

$T = \frac{2\pi}{\left(\frac{5}{\sqrt{3}}\right)^{\frac{1}{2}}}$

$= 3.6780704 45$

$\doteqdot 3.7\,s$

(ii) $v = r\times\left(\frac{5}{\sqrt{3}}\right)^{\frac{1}{2}}$

$\sin 30° = \frac{r}{4}$

$\therefore r = 2$

$v = 3.39808849$

$\doteqdot 3\text{ ms}^{-1}$

(iii) $h = \frac{10}{\left(3\times\frac{5}{\sqrt{3}}\right)^{\frac{3}{2}}}$

$= 0.3923774609$

$\cos\theta = \frac{0.3\ldots}{4}$

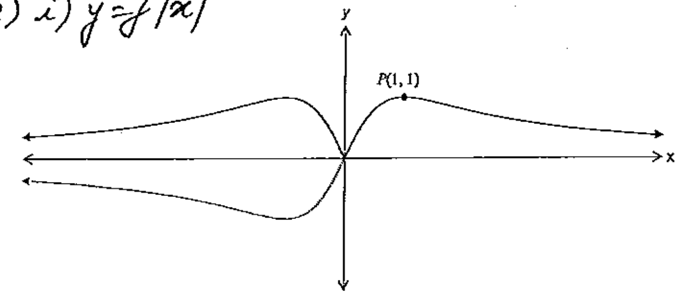
$\therefore \theta = 84°\ 22'\ 13.99''$

$\doteqdot 84°$

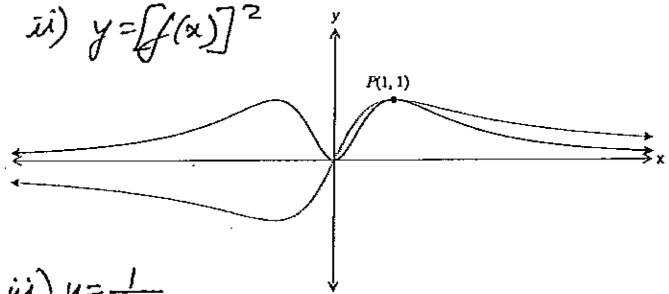
when the speed is increased h decreases while r increases

Question 3.

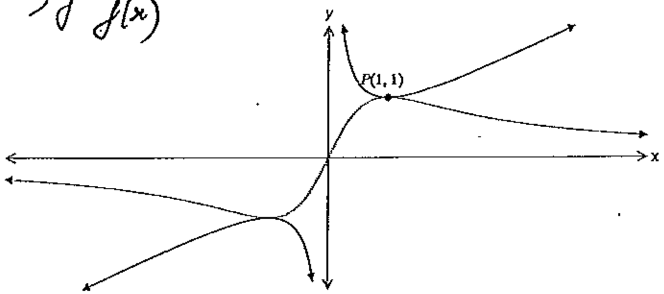
a) i) $y = f|x|$



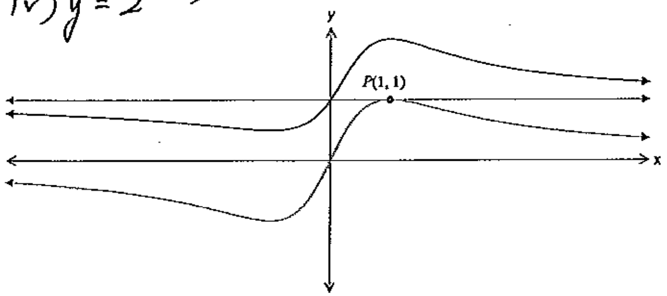
ii) $y = [f(x)]^2$



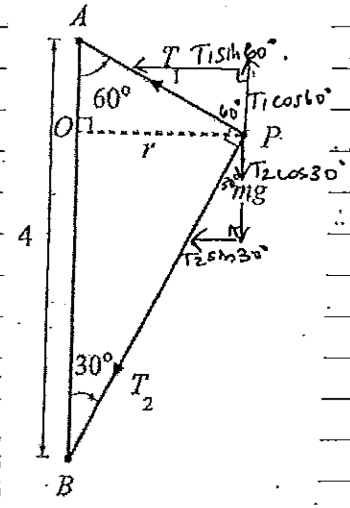
iii) $y = \dfrac{1}{f(x)}$



iv) $y = 2^{f(x)}$



** Students need to pay more attention to the end branches (extremities) of the graphs, as $x \to \pm\infty$.



i) In ΔAPB: $\sin 30° = \dfrac{AP}{4}$

$AP = 4\sin 30° = 2$ ✓

In ΔAOP: $\sin 60° = \dfrac{OP}{AP}$

$OP = AP\sin 60°$

$= 2 \times \dfrac{\sqrt{3}}{2}$ ✓

$OP = \sqrt{3}$ m.

ii)

$r = \sqrt{3}$ m.

$\omega = 30$ rev/min

$= 30 \times 2\pi \times \dfrac{1}{60}$ rad/sec

$= \pi$ rad/sec.

H: $T_1 \sin 60° + T_2 \sin 30° = mr\omega^2$

$\dfrac{\sqrt{3}}{2}T_1 + \dfrac{1}{2}T_2 = 4 \times \sqrt{3} \times (\pi)^2$ ✓

$\therefore \sqrt{3}T_1 + T_2 = 8\sqrt{3}\pi^2 \quad \ldots (1)$

V: $T_1 \cos 60° - T_2 \cos 30° = mg$

$\dfrac{1}{2}T_1 - \dfrac{\sqrt{3}}{2}T_2 = 4g$ ✓

$T_1 - \sqrt{3}T_2 = 8g \quad \ldots (2)$

Solving T_1 and T_2 simultaneously:

From (1): $T_2 = 8\sqrt{3}\pi^2 - \sqrt{3}T_1$ ... (3)

(2) → (3): $T_1 + \sqrt{3}\left(8\sqrt{3}\pi^2 - \sqrt{3}T_1\right) = 8g$

$T_1 - 24\pi^2 + 3T_1 = 8g$

$4T_1 = 8g + 24\pi^2$ ✓

$\therefore T_1 = 2(g + 3\pi^2)$ N

Substituting this into (2):

$2g + 6\pi^2 - \sqrt{3}T_2 = 8g$

$\sqrt{3}T_2 = 6\pi^2 - 6g$

$T_2 = \frac{6}{\sqrt{3}}(\pi^2 - g)$ ✓

$= 2\sqrt{3}(\pi^2 - g)$ N

** students lost marks if they did not show full derivation of the given results

Q4. 2015 Ext 2 Task 1

(a) (i) $f(x) = \dfrac{x^2+2x+4}{x^2-x-6}$

$= \dfrac{x^2+2x+4}{(x-3)(x+2)}$

The first part of this question was well done by most students.

$x \neq 3, -2$

$\lim_{x\to\infty} f(x) = 1^+ \quad \lim_{x\to-\infty} f(x) = 1^-$

Asymptotes - Vertical $x = 3, x = -2$.

Horizontal $y = 1$.

(ii) Intercepts when $x = 0$, $y = -\frac{2}{3}$.

$y = 0$ when $x^2+2x+4 = 0$.

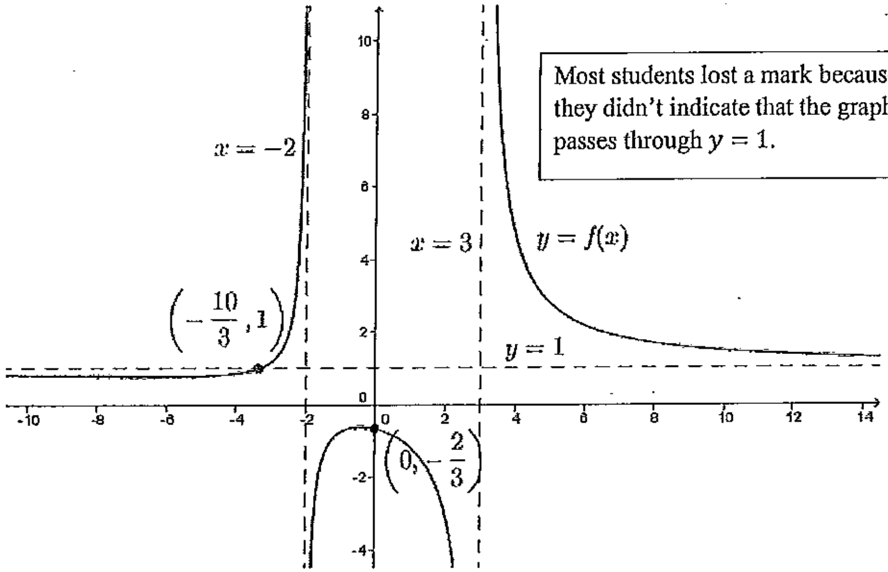
x^2+2x+4 is positive definite.

$\dfrac{x^2+2x+4}{x^2-x-6} = 1$

$x^2+2x+4 = x^2-x-6$.

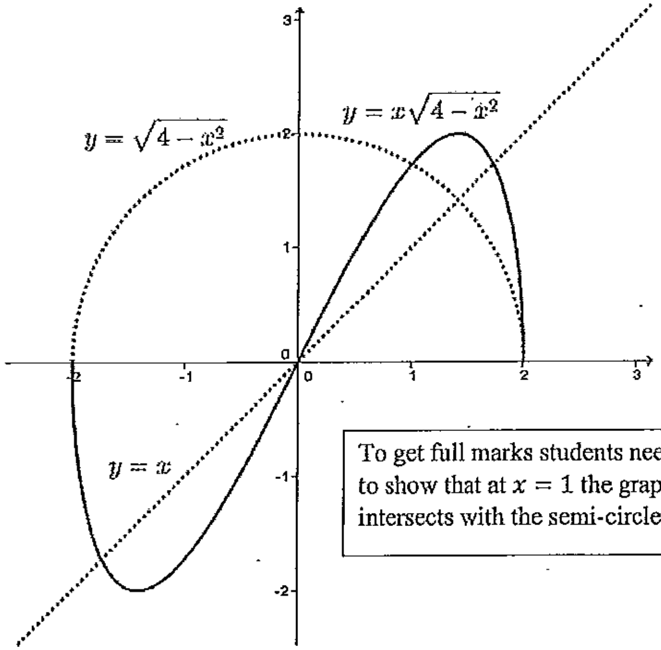
$3x = -10$

$x = -\frac{10}{3}$



Most students lost a mark because they didn't indicate that the graph passes through $y = 1$.

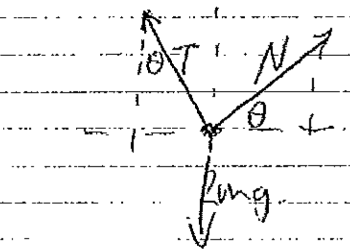
(b)

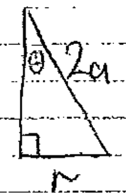


To get full marks students needed to show that at $x = 1$ the graph intersects with the semi-circle.

(c)

(i)





$\sin\theta = \frac{r}{2a}$

$r = 2a\sin\theta$

Horizontal

$T\sin\theta - N\cos\theta = 2mr\omega^2$

Vertical

$T\cos\theta + N\sin\theta = 2mg$ (B)

$T\sin\theta - N\cos\theta = 4ma\sin\theta\,\omega^2$ (A)

(A) $\times \sin\theta$

$T\sin^2\theta - N\cos\theta\sin\theta = 4ma\sin^2\theta\,\omega^2$ (C)

(B) $\times \cos\theta$

$T\cos^2\theta + N\cos\theta\sin\theta = 2mg\cos\theta$. (D)

(C) + (D).

$T = 4ma\sin^2\theta\,\omega^2 + 2mg\cos\theta$.

$= 2m(g\cos\theta + 2a\sin^2\theta\,\omega^2)$

Some students lost a mark for not <u>showing</u> that $r = 2a\sin\theta$. But otherwise this part of the question was well done by most students.

(ii) B $\times \sin\theta$ − A $\cos\theta$.

$N = 2mg\sin\theta - 4ma\omega^2\sin\theta\cos\theta$.

$N = 2m(g\sin\theta - 2a\omega^2\sin\theta\cos\theta)$

(iii) Particle is on the surface of the cone so long as there is a normal reaction force.

$N > 0$

$g\sin\theta - 2a\omega^2\sin\theta\cos\theta > 0$.

$g > 2a\omega^2\cos\theta$ since $\sin\theta > 0$ for $0 < \theta \le \frac{\pi}{2}$.

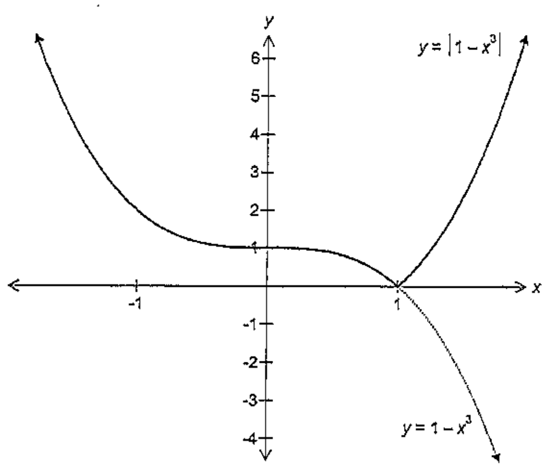
$\omega^2 < \frac{g}{2a\cos\theta}$ since $\cos\theta > 0$ for $0 < \theta \le \frac{\pi}{2}$.

$\omega < \left(\frac{g}{2a\cos\theta}\right)^{\frac{1}{2}}$ Since both side are > 0.

Mathematics Extension 2 HSC Task 1 Nov 2015 Solutions

QUESTION 1

(a)(i)



**The concavity of this graph is a key element. Some students reflected the graph of $y = 1 - x^3$ but did not identify the correct shaping when $x > 1$.
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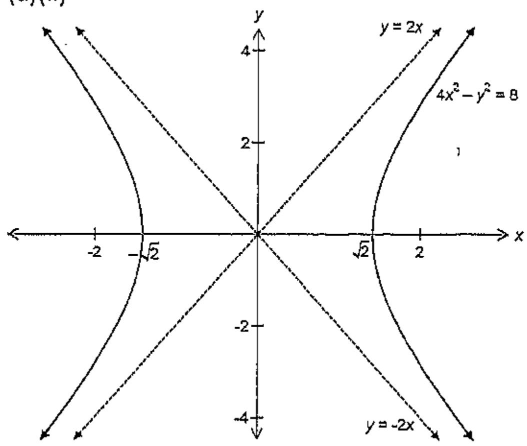

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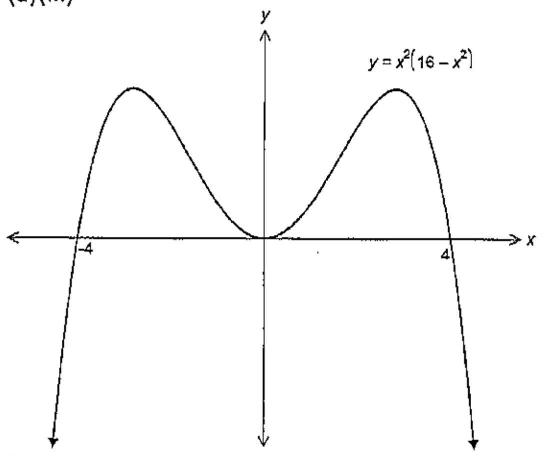
Some students did not correctly identify the equations of the asymptotes.

(a)(iii)



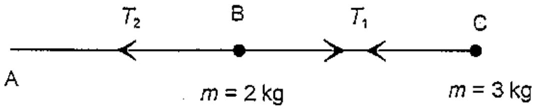
**A range of students chose to approach this by sketching the product of $y = x^2$ and $y = 16 - x^2$. Recognising the function as a polynomial is easier.

(b) $m = 3\,\text{kg}$; $r = 1.2\,\text{m}$; $v = 5\,\text{m/s}$

$T = \dfrac{mv^2}{r} = \dfrac{3 \times 5^2}{1.2}$ $\quad \therefore T = 62.5\ \text{N}$

Generally well done though some students forgot to convert the radius into the correct units (m) or incorrectly identified 5 as the ***angular* speed.

(c)(i) $\omega = 5\ \text{rad/s}$; $T = mr\omega^2$



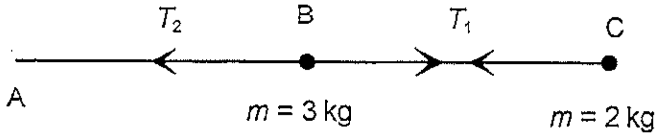
For 3 kg mass : $T_1 = 3 \times 2 \times 5^2 = 150\ \text{N}$

For 2 kg mass : $T_2 - T_1 = 2 \times 1 \times 5^2 = 50\ \text{N}$

Breaking strain : $T_2 = 50 + T_1 = 200\ \text{N}$

**Some solutions did not demonstrate a clear understanding of the relationship between the forces including the tension forces, including that the greatest tension occurs closest to the centre of the system. Some students incorrectly added T_1 and T_2.

(c)(ii) Maximum angular speed $\omega = ?$ when $T_2 = 200\ \text{N}$; $T = mr\omega^2$



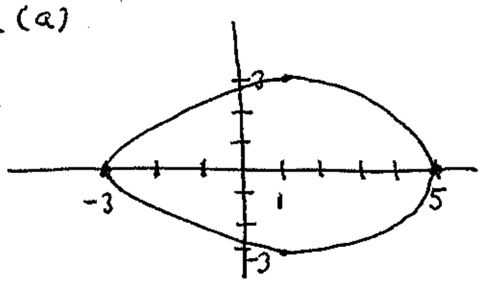
For 2 kg mass : $\quad T_1 = 2 \times 2 \times \omega^2 = 4\omega^2$

For 3 kg mass : $\quad T_2 - T_1 = 3 \times 1 \times \omega^2 = 3\omega^2 \quad \therefore T_2 = 3\omega^2 + T_1 = 7\omega^2$

$7\omega^2 = 200 \quad \therefore \omega = \sqrt{\dfrac{200}{7}} \approx 5.35\ \text{rad/s}$

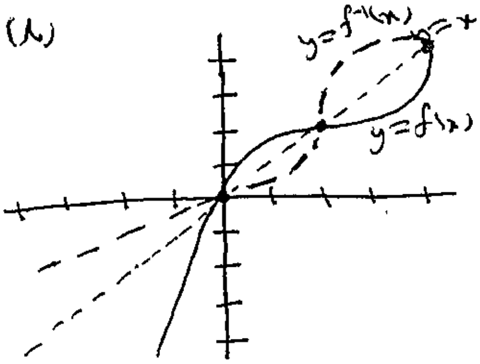
**Using the answer from (i), students needed to calculate the value of the angular speed ω.

2 (a)



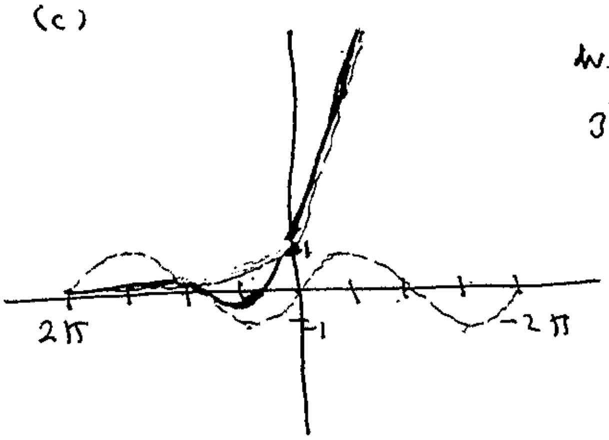
the y intercepts are <u>not</u> ± 3

(b)



$f(x)$ and $f^{-1}(x)$ intersect in 3 places on $y = x$

(c)



when $x > 0$ 3^x dominates

(d)(i) $\cos 30^\circ = \frac{h}{4}$

$\frac{\sqrt{3}}{2} = \frac{h}{4}$

$h = 2\sqrt{3}$

$2\sqrt{3} = \frac{g}{w^2}$

$w^2 = \frac{10}{2\sqrt{3}}$

$w = \left(\frac{5}{\sqrt{3}}\right)^{\frac{1}{2}}$

$T = \frac{2\pi}{\left(\frac{5}{\sqrt{3}}\right)^{\frac{1}{2}}}$

$= 3.6980704 45$

$\doteqdot 3.70$

(ii) $v = r \times \left(\frac{5}{\sqrt{3}}\right)^{\frac{1}{2}}$

$\sin 30^\circ = \frac{r}{4}$

$\therefore r = 2$

$v = 3.39808849$

$\doteqdot 3 \text{ m s}^{-1}$

(iii) $h = \frac{10}{\left(3 \times \frac{5}{\sqrt{3}}\right)^{\frac{3}{2}}}$

$= 0.392377 4609$

$\cos \theta = \frac{0.3\ldots}{4}$

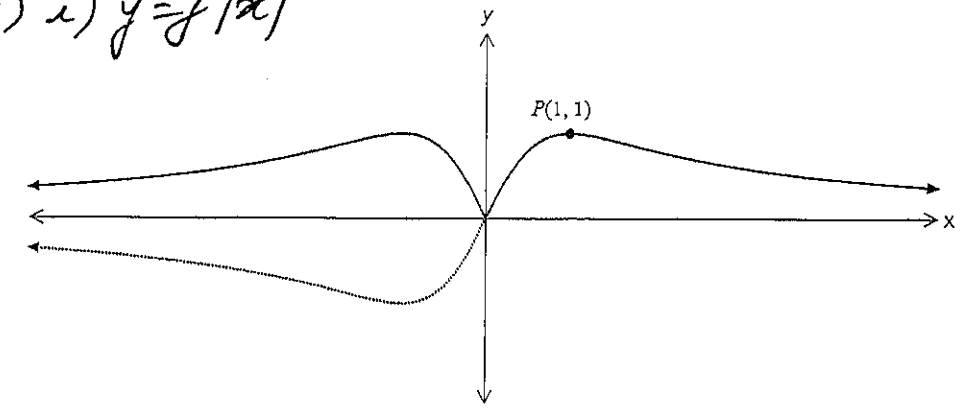
$\therefore \theta = 84^\circ\ 22'\ 13.99''$

$\doteqdot 84^\circ$

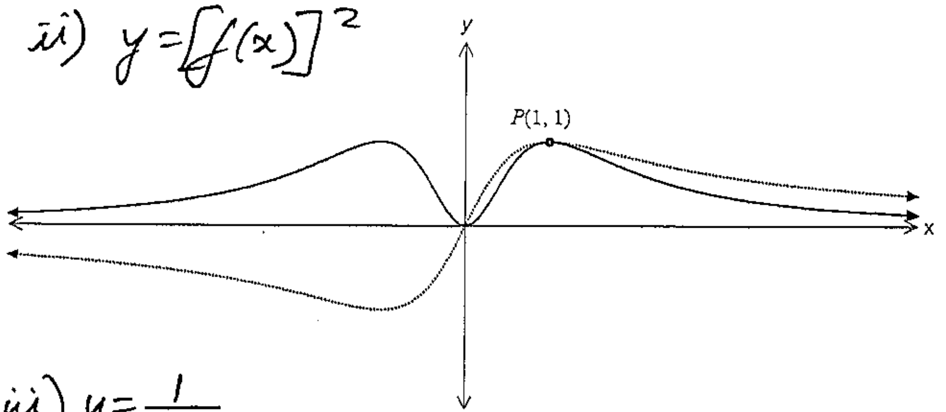
when the speed is increased h decreases while r increases

Question 3

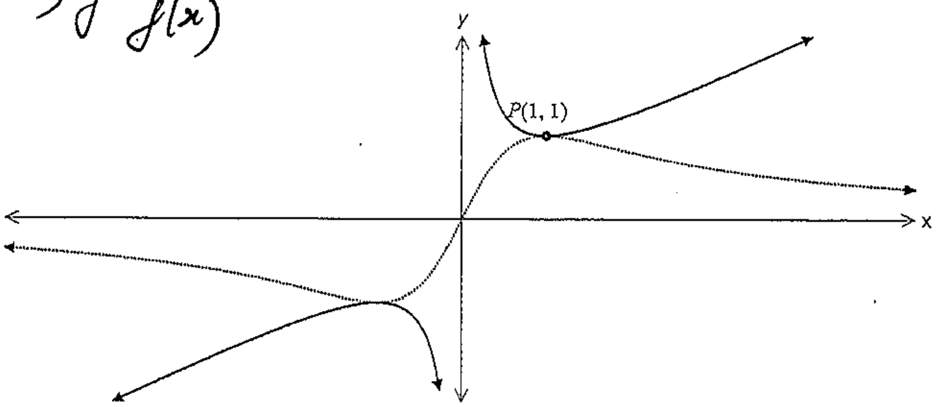
a) i) $y = f|x|$



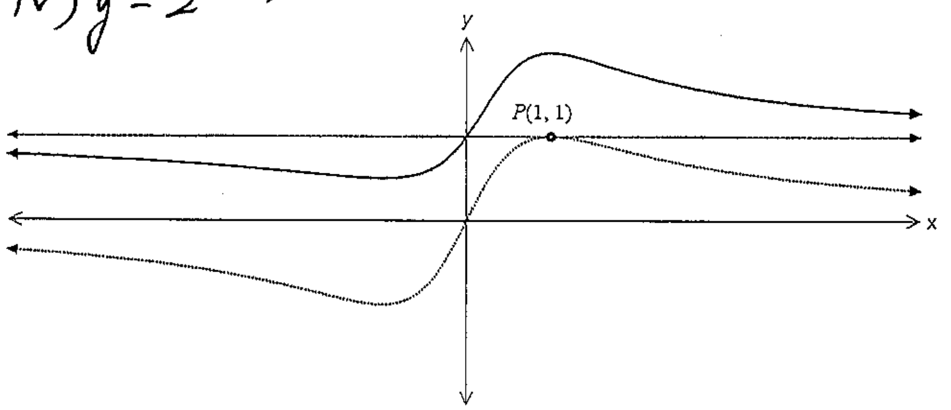
ii) $y = [f(x)]^2$



iii) $y = \frac{1}{f(x)}$

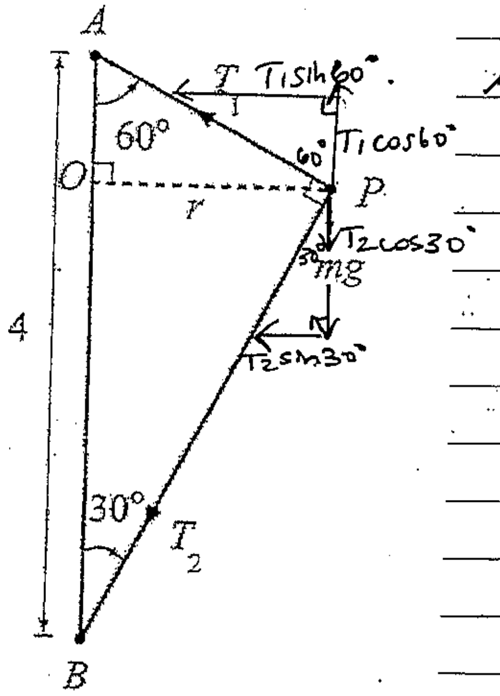


iv) $y = 2^{f(x)}$



** Students need to pay more attention to the end branches (extremities) of the graphs, as $x \to \pm\infty$.

/ /



i) In ΔAPB : $\sin 30° = \frac{AP}{4}$

$AP = 4\sin 30° = 2$

In ΔAOP. $\sin 60° = \frac{OP}{AP}$

$OP = AP\sin 60°$

$= 2 \times \frac{\sqrt{3}}{2}$

$OP = \sqrt{3}$ m.

ii)

$r = \sqrt{3}$ m.

$\omega = 30$ rev/min

$= 30 \times 2\pi \times \frac{1}{60}$ rad/sec

$= \pi$ rad/sec.

H : $T_1\sin 60° + T_2\sin 30° = mr\omega^2$

$\frac{\sqrt{3}}{2}T_1 + \frac{1}{2}T_2 = 4 \times \sqrt{3} \times (\pi)^2$

$\therefore \sqrt{3}T_1 + T_2 = 8\sqrt{3}\pi^2 \quad \dots (1)$

V: $T_1\cos 60° - T_2\cos 30° = mg$

$\frac{1}{2}T_1 - \frac{\sqrt{3}}{2}T_2 = 4g$

$T_1 - \sqrt{3}T_2 = 8g \quad \dots (2)$

/ /

Solving T_1 and T_2 simultaneously:

From 1) $T_2 = 8\sqrt{3}\pi^2 - \sqrt{3}T_1 \quad \ldots (3)$

(2) → (3) : $T_1 - \sqrt{3}\left(8\sqrt{3}\pi^2 - \sqrt{3}T_1\right) = 8g$

$$T_1 - 24\pi^2 + 3T_1 = 8g$$

$$4T_1 = 8g + 24\pi^2 \checkmark$$

$$\therefore T_1 = 2\left(g + 3\pi^2\right) N$$

Substituting this into (2) :

$$2g + 6\pi^2 - \sqrt{3}T_2 = 8g$$

$$\sqrt{3}T_2 = 6\pi^2 - 6g$$

$$T_2 = \frac{6}{\sqrt{3}}\left(\pi^2 - g\right) \checkmark$$

$$= 2\sqrt{3}\left(\pi^2 - g\right) N$$

** students lost marks if they did not show full derivation of the given results

Q4. 2015 Ext 2 Task 1

(a) (i) $f(x) = \dfrac{x^2+2x+4}{x^2-x-6}$

The first part of this question was well done by most students.

$= \dfrac{x^2+2x+4}{(x-3)(x+2)}$.

$x \neq 3, -2$

$\lim_{x\to\infty} f(x) = 1^+ \quad \lim_{x\to-\infty} f(x) = 1^-$

Asymptotes - Vertical $x=3, x=-2$.

Horizontal $y=1$.

(ii) Intercepts when $x=0, y=-\frac{2}{3}$

$y=0$ when $x^2+2x+4=0$.

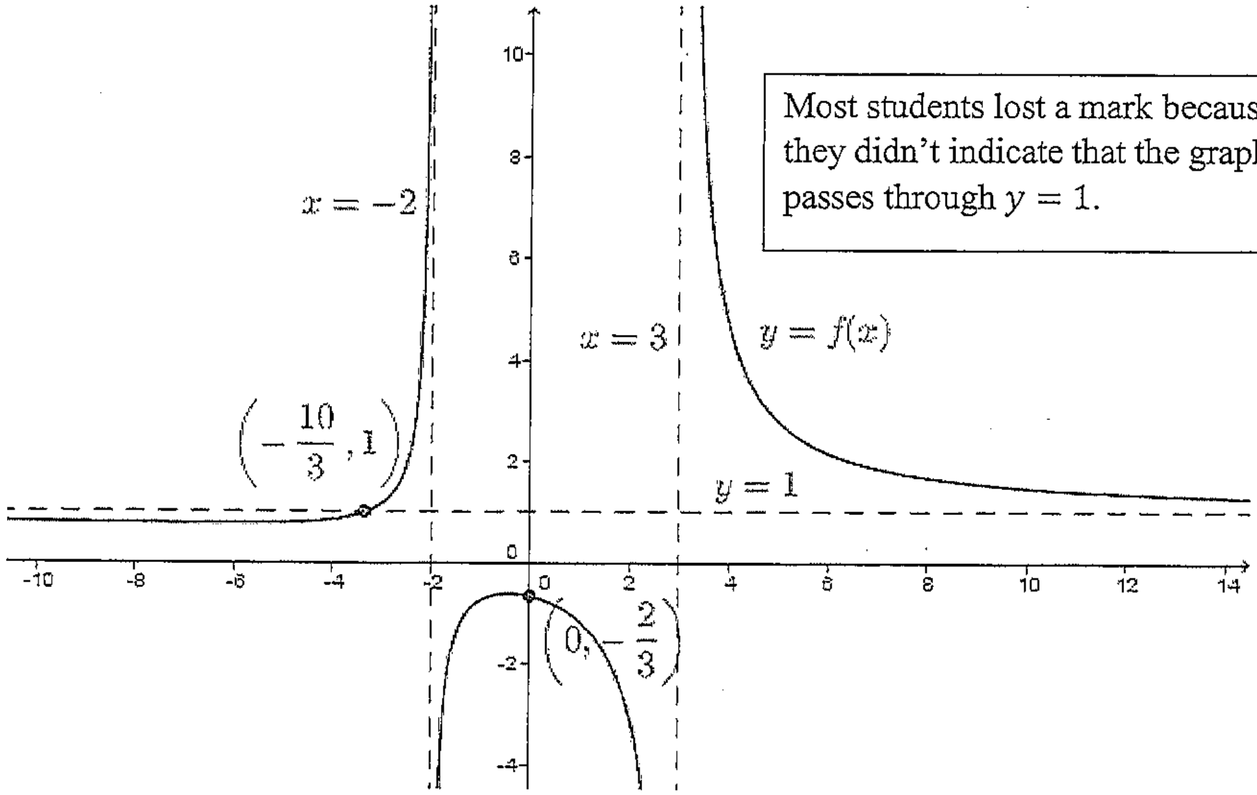
x^2+2x+4 is positive definite.

$\dfrac{x^2+2x+4}{x^2-x-6} = 1$

$x^2+2x+4 = x^2-x-6$.

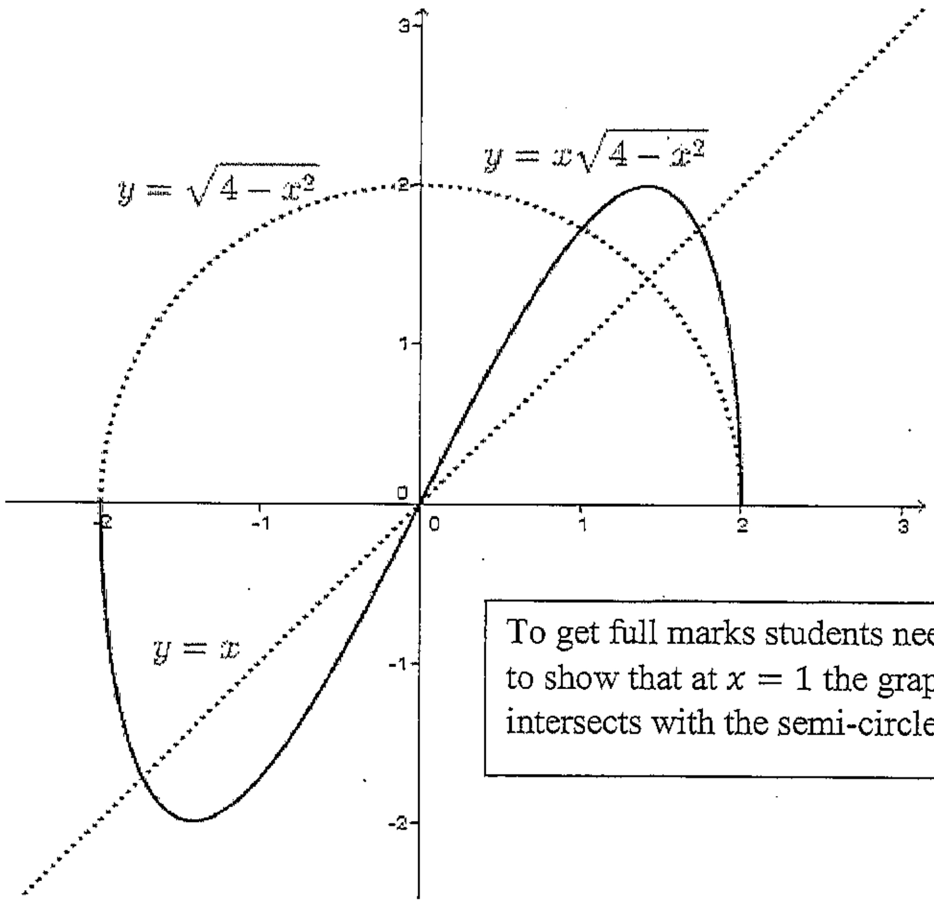
$3x = -10$

$x = -\frac{10}{3}$



Most students lost a mark because they didn't indicate that the graph passes through $y = 1$.

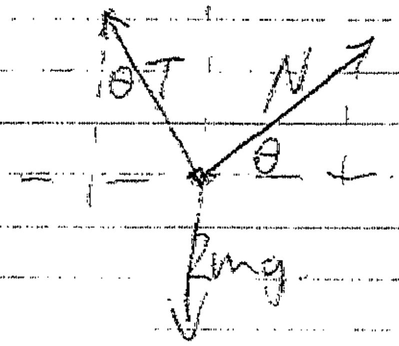
(b)

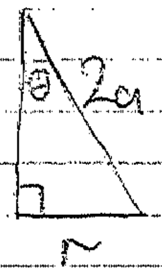


To get full marks students needed to show that at $x = 1$ the graph intersects with the semi-circle.

(c)

(i)





$\sin\theta = \frac{r}{2a}$

$r = 2a\sin\theta$

Horizontal

$T\sin\theta - N\cos\theta = 2mr\omega^2$

$T\sin\theta - N\cos\theta = 4ma\sin\theta\,\omega^2$ (A)

(A) $\times \sin\theta$

$T\sin^2\theta - N\cos\theta\sin\theta = 4ma\sin^2\theta\,\omega^2$ (C)

Vertical

$T\cos\theta + N\sin\theta = 2mg$ (B)

(B) $\times \cos\theta$

$T\cos^2\theta + N\cos\theta\sin\theta = 2mg\cos\theta$. (D)

(C) + (D).

$T = 4ma\sin^2\theta\,\omega^2 + 2mg\cos\theta$.

$= 2m(g\cos\theta + 2a\sin^2\theta\,\omega^2)$

Some students lost a mark for not <u>showing</u> that $r = 2a\sin\theta$. But otherwise this part of the question was well done by most students.

(ii) $B \times \sin\theta - A \cos\theta$.

$N = 2mg\sin\theta - 4ma\omega^2 \sin\theta\cos\theta$.

$N = 2m(g\sin\theta - 2a\omega^2 \sin\theta\cos\theta)$

(iii) Particle is on the surface of the cone so long as there is a normal reaction force.

$N > 0$

$g\sin\theta - 2a\omega^2 \sin\theta\cos\theta > 0$.

$g > 2a\omega^2\cos\theta$ since $\sin\theta > 0$ for $0 < \theta < \frac{\pi}{2}$.

$\omega^2 < \dfrac{g}{2a\cos\theta}$ since $\cos\theta > 0$ for $0 < \theta < \frac{\pi}{2}$.

$\omega < \left(\dfrac{g}{2a\cos\theta}\right)^{\frac{1}{2}}$ since both side are > 0.