1. If $z = 1 - 3i$ and $w = -1 + i$ find:
   (a) zw
   (b) $\arg(-2w)$ $-\pi < \theta \leq \pi$

2. (a) Prove algebraically that $\overline{(z^2)} = (\overline{z})^2$.
   (b) If the complex number z is a solution of the quadratic equation $ax^2 + bx + c = 0$, where a, b and c are real numbers, prove that $\overline{z}$ is also a solution.

3. (a) Find both square roots of $8 + 6i$, in the form $x + iy$.
   (b) Use part (a) to help solve the quadratic equation $z^2 + (2 + 4i)z - 11 - 2i = 0$.

4. Given $z = 1 + i\sqrt{3}$ and $w = 1 + i$:
   (a) Evaluate $\dfrac{z}{w}$.
   (b) Plot z, w and $\dfrac{z}{w}$ on the argand diagram.
   (c) Express z and w in modulus-argument form.
   (d) Show that $\dfrac{z}{w} = \sqrt{2}\,\mathrm{cis}(\frac{\pi}{12})$ and hence find the exact value of $\cos(\frac{\pi}{12})$.

5. Sketch and describe the locus of those points z which satisfy:
   (a) $|z - i| < 2$ and $\dfrac{\pi}{3} \leq \arg(z - i) \leq \pi$
   (b) $\arg\left(\dfrac{z-2}{z+2i}\right) = \dfrac{3\pi}{4}$ — show clearly any intersections with the real and imaginary axis.

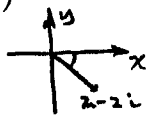
6. (a) Find the cartesian equation of the locus of z if $w = \dfrac{z+2}{z+i}$ is purely real. [HINT: If w is real then $\mathrm{Im}(w) = 0$.]
   (b) Hence sketch the locus.

DNW SGS March 2002

$$zw = (1-3i)(-1+i)$$
$$= -1+3+3i+i$$
$$= 2+4i$$

b) $\arg(-2w) = \arg(2-2i)$

$= -\frac{\pi}{4}$



2. a) LHS $= \overline{(x+iy)^2}$

$= \overline{(x^2-y^2+2xyi)}$

$= x^2-y^2-2xyi$

RHS $= (x-iy)^2$

$= x^2-y^2-2xyi$

$=$ LHS #

b) Since z is a solution

$az^2+bz+c=0$

Take conjugates of both sides

$\overline{(az^2+bz+c)} = 0$

$\therefore \overline{az^2} + \overline{bz} + \overline{c} = 0$

so $a\overline{(z^2)} + b\overline{z} + c = 0$

hence $a(\overline{z})^2 + b(\overline{z}) + c = 0$ by (a)

ie $\overline{z}$ is a soln also.

3. a) $(x+iy)^2 = 8+6i$

so $x^2-y^2=8$ and $2xy=6$

solve simultaneously to get

$x^2 - \frac{9}{x^2} = 8$

$(x^2)^2 - 8x^2 - 9 = 0$

$(x^2-9)(x^2+1) = 0$

so $x = \pm 3$ (since $x \in \mathbb{R}$)

and $y = \pm 1$

roots are $3+i$ and $-3-i$

b) $z = \dfrac{-(2+4i) \pm \sqrt{\Delta}}{2}$

where $\Delta = (2+4i)^2 - 4(-11-2i)$

$= 4-16+16i+44+8i$

$= 32+24i = 4(8+6i)$

so $z = -1-2i \pm \sqrt{8+6i}$

$= -1-2i+3+i$ or

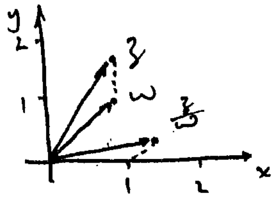
$-1-2i-3-i$

$= 2-i$ or $-4-3i$

4. a) $\dfrac{z}{w} = \dfrac{1+i\sqrt{3}}{1+i} \times \dfrac{1-i}{1-i}$

$= \dfrac{1+\sqrt{3}+i(\sqrt{3}-1)}{2}$

b)



c) $z = 1+i\sqrt{3}$

$= 2 \operatorname{cis} \frac{\pi}{3}$

$w = 1+i$

$= \sqrt{2} \operatorname{cis} \frac{\pi}{4}$

d) $\dfrac{z}{w} = \dfrac{2}{\sqrt{2}} \operatorname{cis}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \sqrt{2} \operatorname{cis} \frac{\pi}{12}$ #.

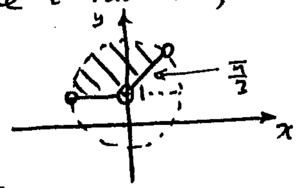
so $\sqrt{2} \cos \frac{\pi}{12} = \dfrac{1+\sqrt{3}}{2}$ (equating real part

$\therefore \cos \frac{\pi}{12} = \dfrac{1+\sqrt{3}}{2\sqrt{2}}$

5. a) inside the circle centre i radius 2,

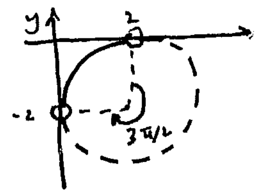
between $\frac{\pi}{3}$ and π,

excluding i.



b) $\arg(z-2) - \arg(z+2i) = \frac{3\pi}{4}$

so the angle in the segment with chord from 2 to $-2i$ is $\frac{3\pi}{4}$. The angle at the centre is $\frac{3\pi}{2}$ & so the centre is at $(2-2i)$ and the radius is 2. The pts 2 and $-2i$ are excluded.



6. a) $w = \dfrac{z+2}{z+i} \times \dfrac{\overline{z}-i}{\overline{z}-i}$

$= \dfrac{|z|^2 + 2\overline{z} - iz - 2i}{|z+i|^2}$

$\operatorname{Im}(w) = 0$ so $\dfrac{-2y-x-2}{|z+i|^2} = 0$

ie $2y+x+2=0$

but $z \neq -i$ so exclude the pt $x=0, y=$

b)