4A 2009 SYDNEY GRAMMAR

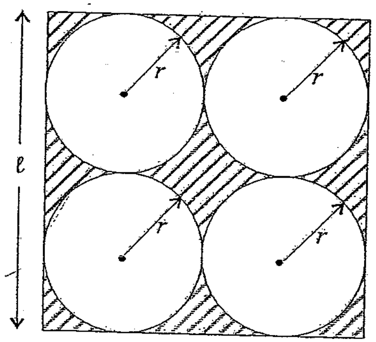
FOURTH FORM CLASS TEST 29/1/09

1 (a) Expand and simplify $(x-y)(x^2+xy+y^2)$.

(b) Factorize the following

i) $16a^2-9b^2$ ii) $2x^2-7x+6$

(c)




Four equal circles, each of radius $r$ cm, are inscribed inside a square of side length $\ell$ cm, as shown in the above diagram.

(i) Write an expression for the length $\ell$ in terms of $r$.

(ii) Hence write an expression for the area of the square in terms of $r$.

(iii) Hence write an expression for the total area of the shaded regions in terms of $r$.

(d) Make $a$ the subject of each formula:

(i) $v=u+at$

(ii) $x=\dfrac{2a+5}{a}$

2 (a) Draw the exact value triangles showing angles and lengths of sides.

(b) Hence write down the exact value of

(i) $\sin 60°$ (ii) $\tan 30°$ (iii) $\cos 45°$.

3 (a) (i) Find the midpoint of the interval joining $A(-1,3)$ and $B(-3,9)$.

(ii) Show that the equation of the perpendicular bisector of the interval joining $A(-1,3)$ and $B(-3,9)$ is $x-3y+20=0$.

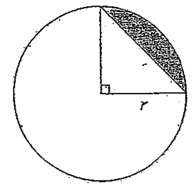
(iii) The perpendicular bisector in part (ii) cuts the $y$-axis at $M$ and the $x$-axis at $N$. Find the area of the triangle $MON$, where $O$ is the origin.

4 From a lighthouse, a ship $A$ bears N24°W and is 5 km away. Another ship $B$ is 8 km away and is in a direction of S66°W.

(a) Draw a clearly labelled diagram to represent this information.

(b) Find the bearing of $A$ from $B$.

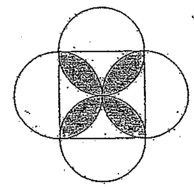
5 (a) (i)




The shaded region in the diagram above is the minor segment formed by a chord that subtends an angle of 90° at the centre of a circle with radius $r$.

Find a formula for the area of this segment in terms of $\pi$ and $r$.

(ii)



Ken was doodling during a geometry lesson and drew the above diagram of four circles with their diameters forming a square.

($\alpha$) Find the total area of the shaded regions.

($\beta$) Hence show that the ratio of the area of the shaded regions to the area of the whole figure is

$$\frac{\pi-2}{\pi+2}.$$

(b) A cylinder has a radius of $r$ centimetres and a height of $h$ centimetres.

(i) Find the volume, in terms of $r$ and $h$, of the cylinder if its radius is increased by 25% and the height is increased by 10%.

(ii) By what percentage has the volume of the cylinder increased?

(c) An outback property, Muttaburra Downs, has an area of 100 000 hectares. The property is rectangular and is 50 km long.

You may use the conversions: $1\,\text{km}^2=100\,\text{ha}$ and $1\,\text{ha}=10\,000\,\text{m}^2$

(i) Find the width of Muttaburra Downs.

(ii) Find the area of an adjacent rectangular property, Billabong Flats, which is 20% shorter but 10% wider than Muttaburra Downs. Give your answer in hectares.

6 (a) A bushwalker covered 34 km in 7 hours. He walked for 3 hours at a speed of $x$ km/h and then for 4 hours at a speed 2 km/h less than his original speed. Form an equation and solve it to find his original speed.

(b) Solve the equation $x + \dfrac{4x}{x^2 - 5} = 0$.

(c) Solve $\sqrt{1+x} = 3 - \sqrt{x}$.

(d) Solve for $x$ and $y$: $\dfrac{x+3}{5} = \dfrac{8-y}{4} = \dfrac{3(x+y)}{8}$

7 (a) If $p^x = 5$ and $p^{x-3} = 20$ find the exact value of $p^{2x-3}$.

(b) Solve $4^{3x-2} = 8\sqrt{2}$.

8 Rationalise the denominator and express in simplest form:

(a) $\dfrac{\sqrt{x-a}}{\sqrt{x+a} - \sqrt{x-a}}$,

(b) $\dfrac{5}{\sqrt[5]{x}}$.

9 Simplify:

(a) $\dfrac{3^{n+4} - 6 \times 3^{n+1}}{3^{n+2} \times 7}$

(b) $\dfrac{5x+2}{2x^2 - 5x - 3} + \dfrac{3x-1}{4x^2 - 1}$

(c) $\dfrac{\frac{1}{x} + \frac{1}{x+1}}{\frac{1}{x} - \frac{1}{x+1}}$

(d) $\dfrac{1}{(a-b)(a-c)} + \dfrac{1}{(b-c)(b-a)} + \dfrac{1}{(c-a)(c-b)}$

10 A goods train is 396 metres long and an express train is 660 metres long. They pass each other in 8 seconds when travelling in opposite directions and the express would take 24 seconds to pass the goods train if they were travelling in the same direction. Find the speed of each train in metres per second.

# FOURTH FORM CLASS TEST

① a) $(x-y)(x^2+xy+y^2)$

$= x^3 - x^2y$
$\quad + x^2y - xy^2$
$\quad + xy^2 - y^3$

$\underline{= x^3 - y^3}$ OR LEARN STANDARD RESULT – DIFFERENCE OF TWO CUBES

b) i) $16a^2 - 9b^2 = (4a)^2 - (3b)^2$ DIFFERENCE OF TWO SQUARES

$\underline{= (4a+3b)(4a-3b)}$

ii) $2x^2 - 7x + 6 = 2x^2 - 3x - 4x + 6$ $\quad S = -7$, $P = 12$

$= x(2x-3) - 2(2x-3)$

$\underline{= (x-2)(2x-3)}$

c) (i) $\underline{l = 4r \text{ cm}}$

(ii) $A = l^2 = (4r)^2$ INCLUDE UNITS!

$\underline{= 16r^2 \text{ cm}^2}$

(iii) Shaded Area $= 16r^2 - 4(\pi r^2)$

$\underline{= 4r^2(4-\pi) \text{ cm}^2}$

d) (i) $v = u + at$

$v - u = at$

$\underline{a = \dfrac{v-u}{t}}$

(ii) $x = \dfrac{2a+5}{a}$

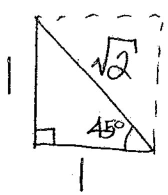
$ax = 2a + 5$

$ax - 2a = 5$

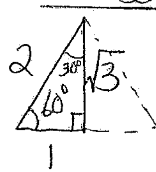
$a(x-2) = 5$

$\underline{a = \dfrac{5}{x-2}}$

② a)




Half a square!




Half an equilateral triangle

b) (i) $\sin 60° = \underline{\dfrac{\sqrt{3}}{2}}$ (ii) $\tan 30° = \dfrac{1}{\sqrt{3}}$ $\left(= \dfrac{\sqrt{3}}{3} \text{ if rationalised}\right)$ (iii) $\cos 45° = \dfrac{1}{\sqrt{2}}$ $\left(= \dfrac{\sqrt{2}}{2}\right)$

③ a) (i) $A(-1,3)$ $B(-3,9)$ $M_{AB} = \left(\dfrac{-1+(-3)}{2}, \dfrac{3+9}{2}\right)$

$\underline{= (-2, 6)}$

(ii) $m_{AB} = \dfrac{9-3}{-3-(-1)} = \dfrac{6}{-2} = -3$

$\therefore m_{\perp} = \dfrac{1}{3}$

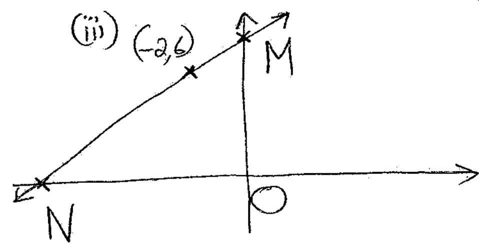
Eqn of perp. bisector $y - y_1 = m(x - x_1)$

$y - 6 = \frac{1}{3}(x - (-2))$

$3y - 18 = x + 2$

$0 = x - 3y + 20$ Q.E.D.

(iii)




to find M $x = 0$ $\therefore$ $0 = -3y + 20$

$3y = 20$

$y = \frac{20}{3}$

$\underline{M(0, 6\tfrac{2}{3})}$

to find N $y = 0$ $\therefore$ $0 = x + 20$

$x = -20$

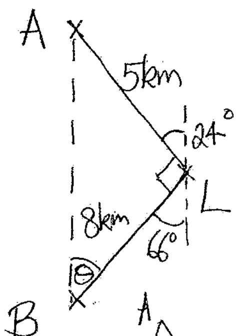
$\underline{N(-20, 0)}$

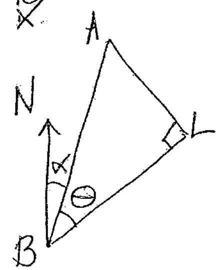
Area $\triangle MNO = \frac{1}{2} \times ON \times OM$

$= \frac{1}{2} \times 20 \times 6\tfrac{2}{3}$

$\underline{= 66\tfrac{2}{3} \text{ u}^2}$

④




$\angle ALB = 180° - (24 + 66)$

$= 90°$

hence $\triangle ALB$ is right-angled

let $\angle ABL = \theta$

$\therefore \tan\theta = \frac{5}{8}$

$\theta = \tan^{-1}\left(\frac{5}{8}\right)$

$\angle NBL = 66°$ Alternate angles // North lines

let $\angle NBA = \alpha$

$\therefore \alpha = 66° - \tan^{-1}\left(\frac{5}{8}\right) \doteqdot 34°00'$

Bearing of A from B is N34E (nearest min)

5 a) (i) Area of Segment = Area of Quadrant − Area of triangle

$$= \frac{\pi r^2}{4} - \frac{1}{2}r^2$$

$$= \frac{r^2}{4}(\pi - 2)$$

(ii) (α) Shaded Region = 8 Segments

$$= 2r^2(\pi - 2) \quad \text{where } r \text{ is radius of circles}$$

(β) Side of Square = $2r$

Area of Square = $(2r)^2$

$= 4r^2$

Area of Whole Figure = Square + Two Circles

$= 4r^2 + 2\pi r^2$

$= 2r^2(2+\pi)$

∴ Ratio of Areas $= \dfrac{2r^2(\pi-2)}{2r^2(\pi+2)}$

$$= \frac{(\pi-2)}{(\pi+2)} \qquad \text{QED}$$

b) Original Dimensions $r, h$

New Dimensions $\dfrac{5r}{4}, \dfrac{11h}{10}$

∴ New Volume $= \pi R^2 H$

$= \pi\left(\dfrac{5r}{4}\right)^2\left(\dfrac{11h}{10}\right)$

$= \pi r^2 h\left(\dfrac{25}{16}\right)\left(\dfrac{11}{10}\right)$

$= \dfrac{275}{160}\pi r^2 h$

$V = \dfrac{55}{32}\pi r^2 h$

(ii) % increase $= \dfrac{\text{increase}}{\text{original volume}} \times 100\% = \dfrac{\frac{23}{32}\cancel{\pi r^2 h} \times 100}{\cancel{\pi r^2 h}}$

$= 71\frac{7}{8}\%$ (= 71.875%)

c) $A = 100\,000$ ha (units are 100m × 100m) $\quad L = 50$ km $= 50\,000$ m $= 500 \times 100$ m.

Rectangle $A = LW$

$100000 = 500 \times W$

$200 = W$

∴ Width is $200 \times 100$ m

$= 20$ km

(ii) $l_2 = \dfrac{8}{10} \times 50$ km $\qquad W_2 = \dfrac{11}{10} \times 20$ km

$l_2 = 40$ km $\qquad W_2 = 22$ km

Area $= 880$ km$^2$

$= 88\,000$ ha

6 a) Let $d_1$ be distance in km at initial ("original") speed

$d_2$ " " " " at slower speed

$d_1 = 3x \qquad d_2 = 4(x-2)$

∴ $3x + 4(x-2) = 34$

$7x - 8 = 34$

$7x = 42$

$x = 6$ km/hr

Original speed was 6km/hr

b) $x + \dfrac{4x}{x^2-5} = 0 \qquad$ (Note $x^2-5 \neq 0$ ie $x \neq \pm\sqrt{5}$)

$(\times\, x^2-5)$ which by definition is non-zero!

$x^3 - 5x + 4x = 0$

$x^3 - x = 0 \qquad$ Cannot divide by $x$ at this stage since $x=0$ a possible solution.

$x(x^2-1) = 0$

$x(x+1)(x-1) = 0$

Solns $x = 0$ or $x = \pm 1$

c) $\sqrt{1+x} = 3-\sqrt{x}$

(sq) (sq)

$1+x = (3-\sqrt{x})^2$

$1+x = 9-6\sqrt{x}+x$

$6\sqrt{x} = 8$

$\sqrt{x} = \frac{8}{6} = \frac{4}{3}$

$x = \frac{16}{9}$

Check LHS $= \sqrt{1+\frac{16}{9}} = \sqrt{\frac{25}{9}} = \frac{5}{3}$

RHS $= 3-\sqrt{\frac{16}{9}} = 3-\frac{4}{3} = \frac{5}{3}$

$\therefore$ Checks

d) Simultaneous eqns

① $\frac{x+3}{5} = \frac{8-y}{4}$

② $\frac{x+3}{5} = \frac{3(x+y)}{8}$

① $\rightarrow 4x+12 = 40-5y$

$\rightarrow 4x+5y = 28$ ③

② $\rightarrow 8x+24 = 15x+15y$

$\rightarrow 7x+15y = 24$ ④

3×③ − ④ $5x = 60$

$x = 12$

$\therefore$ sub into ③

$48+5y = 28$

$5y = -20$

$y = (-4)$

Check

1st fraction $\frac{12+3}{5} = \frac{15}{5} = 3$

2nd $\frac{8-(-4)}{4} = \frac{12}{4} = 3$

3rd $\frac{3(12+(-4))}{8} = 3$

$\therefore$ Checks

⑦ a) $p^x = 5$ ①

$p^{x-3} = 20$ ②

①×② $p^x \times p^{x-3} = 5\times 20$

$p^{2x-3} = 100$

b) $4^{3x-2} = 8\sqrt{2}$

$(2^2)^{3x-2} = 2^3\times 2^{\frac{1}{2}}$

$2^{6x-4} = 2^{7/2}$

equate indices

$6x-4 = \frac{7}{2}$

$6x = \frac{15}{2}$

$x = \frac{15}{12}$

$x = 5/4$

Check

LHS $= 4^{3\times\frac{5}{4}-2} = 4^{(\frac{15}{4}-\frac{8}{4})} = 4^{7/4} = 2^{7/2}$

RHS $= 2^{7/2}$ $\therefore$ Checks

) a) $\frac{\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}} \frac{(\times\sqrt{x+a}+\sqrt{x-a})}{(\times\sqrt{x+a}+\sqrt{x-a})}$

Denominator is perfect square

$= \frac{\sqrt{(x-a)}\left[\sqrt{x+a}+\sqrt{x-a}\right]}{(x+a)-(x-a)}$

$= \frac{\sqrt{x^2-a^2}+(x-a)}{2a}$

b) $\frac{5}{\sqrt[5]{x}} = \frac{5}{x^{1/5}}\frac{(\times x^{4/5})}{(\times x^{4/5})}$

$= \frac{5x^{4/5}}{x}$

$= \frac{5\sqrt[5]{x^4}}{x}$

) a) $\frac{3^{n+4}-6\times 3^{n+1}}{3^{n+2}\times 7} = \frac{3^2\times 3^{n+2}-2\times 3^{n+2}}{7\times 3^{n+2}}$

Cancel common factor $3^{n+2}$

$= \frac{3^2-2}{7}$

$= 1$

b) $\frac{5x+2}{2x^2-5x-3} + \frac{3x-1}{4x^2-1}$

$= \frac{5x+2}{(2x+1)(x-3)} + \frac{3x-1}{(2x+1)(2x-1)}$

$= \frac{(5x+2)(2x-1)+(3x-1)(x-3)}{(2x+1)(2x-1)(x-3)} \leftarrow$ LCD.

$= \frac{10x^2+4x-5x\cancel{-5}-2+3x^2-x-9x+3}{(2x+1)(2x-1)(x-3)}$

$= \frac{13x^2-11x\cancel{-2}+1}{(2x+1)(2x-1)(x-3)}$

$= \cancel{\frac{(13x+2)(x-1)}{(2x+1)(2x-1)(x-3)}}$

c) $\frac{\frac{1}{x}+\frac{1}{x+1}}{\frac{1}{x}-\frac{1}{x+1}} = \frac{\frac{x+1+x}{x(x+1)}}{\frac{x+1-x}{x(x+1)}}$

DENOMINATORS ARE COMMON FACTOR SO CANCEL!

$= \frac{2x+1}{1}$

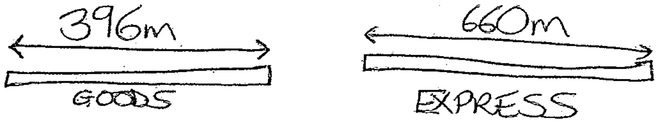
$= 2x+1$

d) $\frac{1}{(a-b)(a-c)} + \frac{1}{(b-c)(b-a)} + \frac{1}{(c-a)(c-b)}$

$= \frac{(b-c)+(c-a)+(a-b)}{(a-b)(a-c)(b-c)}$

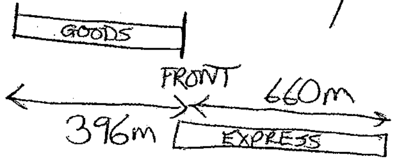
$= 0$

(10)




Let speed of goods train be $g$ m/s.
" " " express " " $e$ m/s. (Where $e > g$ by defn)

Opposite directions - relative speed is $g+e$
Same directions - relative speed is $(e-g)$
ie Goods train stationary




Travels 1056m in
8s at speed $(g+e)$
24s at speed $(e-g)$

$$\text{speed} = \frac{\text{dist}}{\text{time}}$$

$$g+e = \frac{1056}{8} \qquad g+e = 132 \quad ①$$

$$e-g = \frac{1056}{24} \qquad e-g = 44 \quad ②$$

① + ② $\quad 2e = 176$

$$e = 88 \text{ m/s}$$

$$g = 132 - 88 = 44 \text{ m/s}$$

<u>The goods train travels at 44m/s and the express at 88m/s.</u>