SYDNEY GRAMMAR SCHOOL HALF-YEARLY EXAMINATION 2004

MATHEMATICS FORM IV

Time allowed: 2 hours **Exam date:** 17th May 2004

Instructions

All questions may be attempted.
All questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.

Collection

Staple all your paper in one bundle.
Write your name, class and master's initials on the front.
Papers will be collected in class sets:

4A: REP	4B: JNC	4C: BDD
4D: JMR	4E: LYL	4F: PKH
4G: GJ	4H: DNW	4I: JCM

Checklist

Writing paper required.
Candidature: 183 boys

Examiner

DNW

QUESTION ONE

(a) What is the radius of the circle with equation $x^2 + y^2 = 9$?

(b) What is the volume of a cube with edge length 5 cm?

(c) Solve $x^2 = 16$.

(d) Calculate the gradient of the line passing through $A(1, 4)$ and $B(2, 2)$.

(e) Write down the gradient of a line parallel with the line $5x - y - 1 = 0$.

(f) Write 9% p.a. as a monthly rate.

(g) Find the simple interest earned if \$1000 is invested for 3 years at 7·5% p.a.

(h) Factorise $2x^2 + 6x$.

(i) Expand $(2x - 5)^2$.

(j) What is the probability of rolling an even number on a six-sided die?

(k) Is the parabola with equation $y = 3 - x^2$ concave up or concave down?

(l) In which two quadrants does the graph of $y = \dfrac{1}{x}$ lie?

QUESTION TWO

(a) Find the equation of the line with gradient $\frac{1}{2}$ and y-intercept -3. Give your answer in general form.

(b) A circle has equation $(x - 2)^2 + (y + 1)^2 = 6$.

(i) Write down the coordinates of the centre C of this circle.

(ii) Write down the radius r of the circle.

(iii) Find the distance PC, where $P = (1, 1)$.

(iv) Hence determine whether P is inside the circle.

(c) (i) Solve $x^2 - 4x - 12 = 0$ by finding factors.

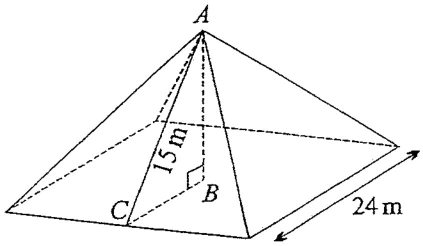
(ii) Solve $x^2 + 3x - 2 = 0$ by the quadratic formula.

(iii) Solve $x^2 - 8x + 10 = 0$ by completing the square.

Exam continues next page ...

QUESTION THREE

(a)



An archæologist took measurements of a small square pyramid that she found in the jungle. The base was 24 metres wide, and the slant height from the middle of one side to the apex was 15 metres, as shown in the diagram above. She discovered that the pyramid was solid, made from stone carried by the local tribe from the nearby quarry.

(i) Find the height of the pyramid.

(ii) How many cubic metres of stone did the tribesmen have to carry to build it?

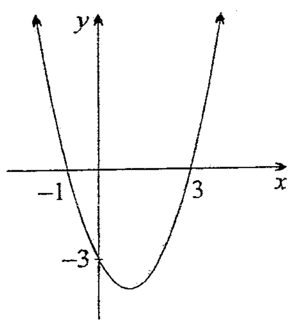
(b) A line ℓ is perpendicular to PQ, where $P = (-2, 1)$ and $Q = (4, 3)$.

(i) Find the gradient of PQ.

(ii) The line ℓ also passes through P. Find its equation.

(iii) Hence find the y-intercept of ℓ.

(c)



The equation of the parabola graphed above is $y = x^2 + bx + c$. Find the values of b and c.

(d) In a game show, Julia is asked to choose one envelope from three coloured envelopes; one pink, one yellow and one blue. Inside each envelope there are four questions, and she selects one of these to answer.

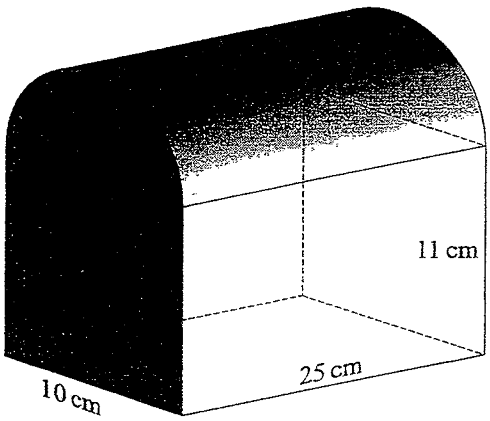
(i) How many questions are there altogether?

(ii) Unknown to Julia, one of the four questions is the same in both the yellow and blue envelopes. What is the probability that she chooses that question?

Exam continues overleaf . . .

QUESTION FOUR

(a)



An old-fashioned worker's lunchbox is in the shape of a rectangular prism surmounted by half a cylinder, as shown in the diagram above. The rectangular prism is 25 cm long, 10 cm wide and 11 cm deep.

(i) What is the radius of the half-cylinder?

(ii) Find the exact volume of the lunchbox in terms of π.

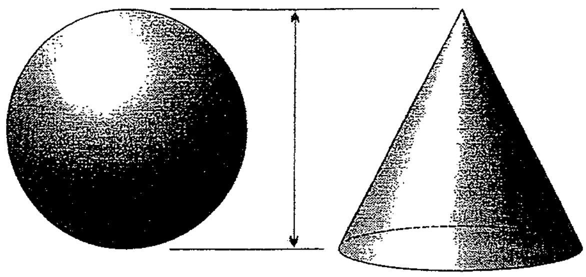
(iii) Find its surface area in square centimetres, correct to two decimal places.

(b) Mr Shimizu has a choice of putting his \$10 000 in an account with the Sydney Savings Bank, which pays compound interest at the rate of 7% p.a. (compounded annually), or in an account with the Kirrawong Investment Society, which pays simple interest at the rate of 8% p.a. Given that the investment is for 5 years, which account should he choose in order to maximise his interest? Justify your answer.

(c) In the game show 'Deal or No Deal', a contestant picks one suitcase from 26 containing hidden cash prizes, the greatest of which is \$200 000. What is the probability that the contestant picks the suitcase containing the \$200 000?

QUESTION FIVE

(a)

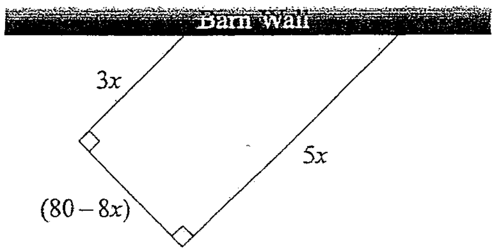


The base radius of the cone and the radius of the sphere in the diagram above are both equal to r. The height of the cone is the same as the diameter of the sphere, as shown.

(i) Show that the volume of the cone is half that of the sphere.

(ii) Show that the surface area of the cone is $\pi r^2(1+\sqrt{5})$.

(b)



A farmer builds a strange pen at an angle to his barn, thus forming a trapezium. The lengths of the parallel sides are $3x$ and $5x$, and the distance between them is $(80-8x)$, as shown in the diagram above. The total length of fencing used is 80 metres, but no fencing is put along the barn wall.

(i) Show that the area of the pen is given by $A = 320x - 32x^2$.

(ii) Find the value of x that makes the area a maximum.

(iii) Hence find the maximum area of the pen.

(iv) Find the length of barn used for this pen.

(c) Five years ago, Jocelyn bought a car worth \$20 000. Jocelyn had to borrow the whole \$20 000 from the credit union, at a flat rate of 7·3% p.a., to be repaid in equal monthly installments over those five years.

(i) The car has since depreciated at the rate of 25% p.a. What is its current value, rounded to the nearest dollar?

(ii) How much has she paid in total to the credit union for her car?

(iii) What was the size of her monthly installment to the credit union?

(iv) Using your answers to parts (i) and (ii), how much more has she paid for the car than it is worth now?

QUESTION SIX

(a) (i) Copy and complete the following table of values for the hyperbola $y = \frac{1}{x} - 1$.

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y						

(ii) Hence sketch the curve $y = \frac{1}{x} - 1$, using a scale of 1 unit = 1 cm on each axis.

(iii) What is the equation of the horizontal asymptote of the hyperbola $y = \frac{1}{x} - 1$?

(b) (i) Graph $y = 2^x$, for $-2 \le x \le 2$, showing the y-intercept and the coordinates of the end-points.

(ii) Graph $y = -x^2 + 2x + 1$ on the same number plane, clearly showing the x-intercept, the y-intercept and the vertex.

(iii) How many times do the two graphs intersect?

QUESTION SEVEN

(a) Five times a positive number is three less than twice its reciprocal. Form an equation and solve it to find the number.

(b) The equation of a particular circle is $x^2 + y^2 - 10y = 144$. Complete the square in y to find its centre and radius.

(c) Consider the quadratic expression $5 - (2x + 1)^2$.

(i) Explain why this expression has a maximum value.

(ii) What value of x gives this maximum value?

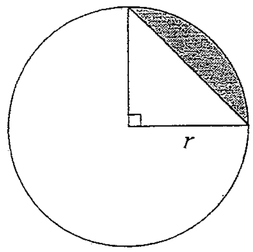
(d) Four cards labelled with the numbers 1, 3, 5 and 7 are put into a hat. One card is drawn and put on the table. A second card is drawn and put on the table to the right of the first to form a two-digit number.

(i) Draw a tree diagram to show all the possible outcomes.

(ii) Hence find the probability that the number formed is divisible by 5.

QUESTION EIGHT

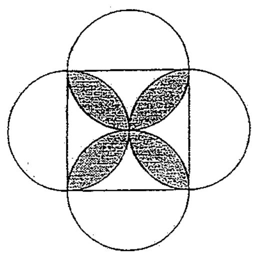
(a) (i)



The shaded region in the diagram above is the minor segment formed by a chord that subtends an angle of $90°$ at the centre of a circle with radius r.

Find a formula for the area of this segment in terms of π and r.

(ii)



Ken was doodling during a geometry lesson and drew the above diagram of four circles with their diameters forming a square.

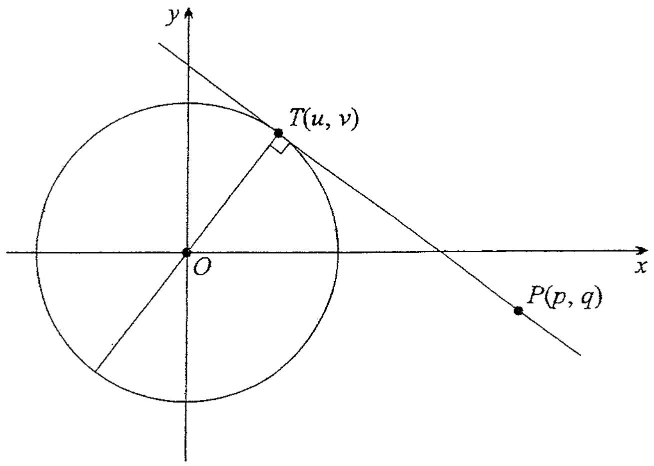
(α) Find the total area of the shaded regions.

(β) Hence show that the ratio of the area of the shaded regions to the area of the whole figure is

$$\frac{\pi - 2}{\pi + 2}.$$

QUESTION EIGHT (Continued)

(b)



The diagram above shows the origin O and the two points $T(u, v)$ and $P(p, q)$ in the number plane. The angle between OT and TP is a right-angle. The circle through T with centre O has equation $x^2 + y^2 = r^2$.

(i) Use the gradients of OT and TP to show that

$$pu + qv = u^2 + v^2 .$$

(ii) Square both sides of this equation and hence show that

$$(q^2 - r^2)m^2 + 2pqm + (p^2 - r^2) = 0 ,$$

where m is the gradient of OT and r is the radius of the circle.

(iii) The equation in part (ii) is a quadratic in m. Find its discriminant $\Delta = b^2 - 4ac$.

(iv) Use Δ to explain why it is not possible for P to be inside the circle.

(v) Now suppose that $r = 5$ and $P = (11, -2)$.

(α) Find the possible values of m.

(β) Find the co-ordinates of T in the case where $m > 0$.

END OF EXAMINATION

SGS Half-yearly 2004 MATHEMATICS FORM IV Solutions

QUESTION ONE

(a) $r = 3$ ☑

(b) $V = 125\,\text{cm}^3$ ☑

(c) $x = 4$ or -4 ☑

(d) gradient $m = \dfrac{2-4}{2-1}$
$= -2$ ☑

(e) gradient $m = 5$ ☑

(f) 9% p.a. $= 0{\cdot}75\%$ per month ☑

(g) Interest $I = 1000 \times \dfrac{7.5}{100} \times 3$
$= \$225$ ☑

(h) $2x^2 + 6x = 2x(x+3)$ ☑

(i) $(2x-5)^2 = 4x^2 - 20x + 25$ ☑

(j) $P(\text{even}) = \frac{3}{6}$
$= \frac{1}{2}$ ☑

(k) The parabola $y = 3 - x^2$ is concave down. ☑

(l) The graph of $y = \dfrac{1}{x}$ lies in the 1st and 3rd quadrants. ☑

Total for Question 1: 12 Marks

QUESTION TWO

(a) Now $y = \frac{1}{2}x - 3$ (gradient-intercept form) ☑
so $\frac{1}{2}x - y - 3 = 0$
hence $x - 2y - 6 = 0$ ☑

(b) (i) $C = (2, -1)$ ☑

(ii) $r = \sqrt{6}$ ☑

(iii) $PC^2 = (1-2)^2 + (1+1)^2$
$= 5$
thus $PC = \sqrt{5}$ ☑

(iv) Since $PC < r$, P is inside the circle. ☑

(c) (i) Now $(x-6)(x+2) = 0$ ☑
hence $x = 6$ or -2 ☑

(ii) Here $b^2 - 4ac = 3^2 - 4 \times 1 \times (-2)$
$= 17$ ☑
thus $x = \dfrac{-3+\sqrt{17}}{2}$ or $\dfrac{-3-\sqrt{17}}{2}$ ☑

(iii) This gives $x^2 - 8x + 16 = 6$
or $(x-4)^2 = 6$ ☑
so $x = 4 + \sqrt{6}$ or $4 - \sqrt{6}$ ☑

Total for Question 2: 12 Marks

QUESTION THREE

(a) (i) The length of BC is 12 m, so
$AB^2 = AC^2 - BC^2$ (by Pythagoras)
$= 225 - 144$
$= 81$
thus $AB = 9$ ☑

(ii) Volume $= \frac{1}{3} \times 24^2 \times 9$ ☑
$= 1728\,\text{m}^3$ ☑

(b) (i) Gradient $m_1 = \dfrac{3-1}{4-(-2)}$
$= \frac{1}{3}$ ☑

(ii) The gradient m_2 of ℓ is given by
$m_2 = \dfrac{-1}{m_1}$
$= -3$. ☑
Using the point-gradient formula:
$y - 1 = -3(x+2)$
or $y = -3x - 5$ ☑

(iii) Thus y-intercept $= -5$ ☑

(c) From the y-intercept we have:

$$c = -3\,.$$

Using the x-intercept at $(-1, 0)$:

$$0 = 1 - b - 3$$

so $\quad b = -2\,.$

Thus $\quad y = x^2 - 2x - 3\,.$

(d) (i) Number of questions $= 4 \times 3$

$$= 12$$

(ii) Number of same questions $= 2$

$$\text{Thus } P(\text{that question}) = \frac{2}{12} = \frac{1}{6}$$

Total for Question 3: 12 Marks

QUESTION FOUR

(a) (i) Radius of cylinder $= 5\,\text{cm}$

(ii) Volume $= \text{prism} + \frac{1}{2} \times \text{cylinder}$

$$= 10 \times 11 \times 25 + \tfrac{1}{2}\pi \times 5^2 \times 25$$

$$= \frac{5500 + 625\pi}{2}\ \text{cm}^3$$

(iii) Surface area $= \text{bottom} + 2 \times \text{front} + 2 \times \text{end} + \frac{1}{2} \times \text{cylinder}$

$$= 10 \times 25 + 2 \times 11 \times 25 + 2 \times 10 \times 11 + \tfrac{1}{2} \times 2 \times \pi \times 5(5 + 25)$$

$$= 1491{\cdot}238\ldots$$

$$\doteqdot 1491{\cdot}24\,\text{cm}^2 \quad \text{(to two decimal place)}$$

(b) For the SSB

$$A = 10\,000(1 + 0.07)^5$$

$$\doteqdot \$14\,025.52$$

For the KIS

$$A = 10\,000 + 10\,000 \times 0.08 \times 5$$

$$= \$14\,000$$

Thus he should invest with SSB which is better by \$25.52.

(c) $$P(\text{prize}) = \frac{1}{26}$$

Total for Question 4: 12 Marks

QUESTION FIVE

(a) (i)

$$V_{sphere} = \tfrac{4}{3}\pi r^3$$

$$V_{cone} = \tfrac{1}{3}\pi r^2 h = \tfrac{2}{3}\pi r^3 = \tfrac{1}{2}V_{sphere}$$

(ii) $S.A. = \pi r(r + s)\,.$

Here $\quad s^2 = r^2 + (2r)^2 \quad$ (by Pythagoras)

thus $\quad s = r\sqrt{5}$

hence $\quad S.A. = \pi r^2(1 + \sqrt{5})$

(b) (i) Using the formula $\dfrac{a+b}{2} \times h$ gives

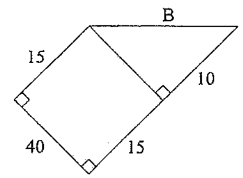
$$A = \frac{5x + 3x}{2} \times (80 - 8x) = 320x - 32x^2\,.$$

(ii) The maximum happens when

$$x = \frac{-320}{2 \times -32} = 5\,\text{m}$$

(iii) Thus $A_{\max} = 320 \times 5 - 32 \times 25$

$$= 800\,\text{m}^2$$



(iv) Let the length of barn be B, then

$$B^2 = 40^2 + 10^2 \quad \text{(by Pythagoras)}$$

thus $\quad B = 10\sqrt{17}$ metres

(c) (i) $$A = 20\,000 \times (1 - 0.25)^5$$

$$\doteqdot \$4746 \quad \text{(to the nearest \$)}$$

(ii) Now $\quad I = 20\,000 \times 0{\cdot}073 \times 5$

$$= \$7300\,.$$

So the amount repaid was

$$A = \$27\,300\,.$$

(iii) The monthly installment M was thus

$$M = \frac{27\,300}{60} = \$455$$

(iv) Difference $= 27\,300 - 4746$

$$= \$22\,554$$

Total for Question 5: 12 Marks

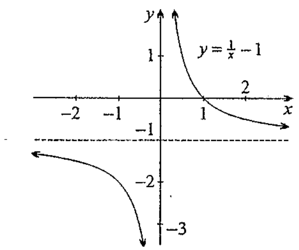
QUESTION SIX

(a) (i)

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y	$-1\frac{1}{2}$	-2	-3	1	0	$-\frac{1}{2}$

[√√]

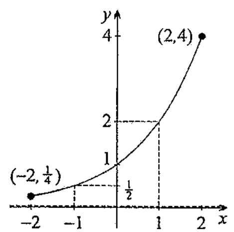
(ii)



shape [√]
x-intercept [√]

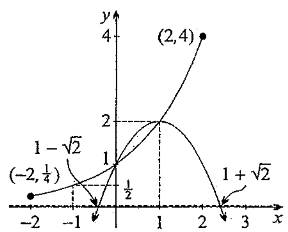
(iii) The asymptote is $y = -1$. [√]

(b) (i)



y-intercept [√]
end-points / shape [√]

(ii)



y-intercept [√]
x-intercepts [√√]
vertex [√]

(iii) The number of intersection points of the two graphs in part (ii) is 2. [√]

Total for Question 6: 12 Marks

QUESTION SEVEN

(a) (i) Let n be the number then,

$$5n = \frac{2}{n} - 3$$ [√]

so $5n^2 + 3n - 2 = 0$

or $(5n-2)(n+1) = 0$ [√]

thus $n = \frac{2}{5}$ or -1. [√]

But $n > 0$ hence

$n = \frac{2}{5}$ [√]

(b) $x^2 + y^2 - 10y + 25 = 169$ [√]

or $x^2 + (y-5)^2 = 13^2$

thus the centre is $(0, 5)$ [√]

and the radius is 13. [√]

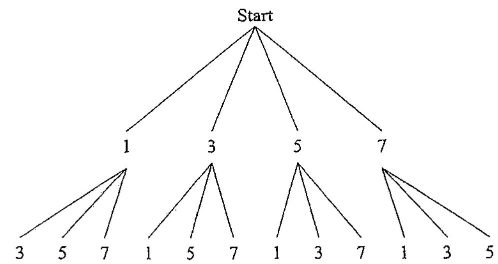
(c) (i) The coefficient of x^2 is $a = -4 < 0$. [√]

[Of course, also accept any valid worded argument.]

(ii) This happens when $2x + 1 = 0$

that is $x = -\frac{1}{2}$ [√]

(d) (i)



Results for first draw [√]
Results for 2nd draw [√]

(ii) $P(\text{divisible by } 5) = \frac{3}{12}$

$= \frac{1}{4}$ [√]

Total for Question 7: 12 Marks

QUESTION EIGHT

(a) (i) Clearly $A_{segment} = A_{sector} - A_{triangle}$

$= \frac{1}{4}\pi r^2 - \frac{1}{2}r^2$. [√]

(ii) (α) The shaded region is made up of 8 segments like that found in part (i) so

$A_{shaded} = 8 \times \left(\frac{1}{4}\pi r^2 - \frac{1}{2}r^2\right)$

$= 2\pi r^2 - 4r^2$ [√]

$= 2r^2(\pi - 2)$.

(β) The total area of the figure is two circles plus a square so:

$A_{total} = 2\pi r^2 + 4r^2$ [√]

$= 2r^2(\pi + 2)$,

thus $ratio = \dfrac{2r^2(\pi - 2)}{2r^2(\pi + 2)}$ [√]

$= \dfrac{\pi - 2}{\pi + 2}$.

(b) (i) Now gradient $OT = \dfrac{v}{u}$

and gradient $TP = \dfrac{q-v}{p-u}$, ☑

so $\dfrac{v}{u} \times \dfrac{q-v}{p-u} = -1$ (perpendicular lines). ☑

Thus $qv - v^2 = -pu + u^2$

or $pu + qv = u^2 + v^2$.

(ii) Squaring:

$$(pu+qv)^2 = (u^2+v^2)^2$$

Divide by u^2 to get:

$$\begin{aligned}(p+qm)^2 &= (u^2+v^2)(1+m^2)\\ &= r^2(1+m^2).\end{aligned}$$ ☑

Expanding,

$$p^2 + 2pqm + q^2m^2 = r^2 + r^2m^2$$

so $(q^2-r^2)m^2 + 2pqm + (p^2-r^2) = 0$.

(iii)

$$\begin{aligned}\Delta &= 4p^2q^2 - 4(q^2-r^2)(p^2-r^2)\\ &= 4p^2q^2 - 4\Big(p^2q^2 - r^2(p^2+q^2) + r^4\Big)\\ &= 4(r^2(p^2+q^2) - r^4)\\ &= 4r^2(p^2+q^2-r^2).\end{aligned}$$ ☑ ☑

(iv) If P is inside the circle then $p^2+q^2 < r^2$ so $p^2+q^2-r^2 < 0$ and the value of Δ is negative. This would give no real solutions for m, and hence P must be outside the circle, or on it. ☑

(v) (α) Substitution gives:

$$\begin{aligned}\Delta &= 4 \times 5^2(11^2+2^2-5^2)\\ &= 100^2.\end{aligned}$$

Hence

$$\begin{aligned}m &= \frac{44 \pm \sqrt{100^2}}{-42}\\ &= -\tfrac{144}{42} \text{ or } \tfrac{56}{42}\\ &= -\tfrac{24}{7} \text{ or } \tfrac{4}{3}.\end{aligned}$$ ☑

(β) For $m = \frac{4}{3}$ and $r = 5$ we have the usual $(3,4,5)$ Pythagorean triad.
Hence $T = (3,4)$. ☑

Total for Question 8: 12 Marks

END OF EXAMINATION