SYDNEY GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT
HALF-YEARLY EXAMINATIONS 2006

FORM IV

MATHEMATICS

Examination date

Monday 22nd May 2006

Time allowed

Two hours

Instructions

All eight questions may be attempted.
All eight questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.

Collection

Write your name, class and master clearly on the front.
Hand in all the writing paper in a single well-stapled bundle.
Keep the printed examination paper and bring it to your next Mathematics lesson.

4A: PKH	4B: LYL	4C: JCM	4D: GJ
4E: JMR	4F: REP	4G: BDD	4H: SJE
4I: DNW	4J: KWM		

Checklist

Writing paper required.
Candidature: 192 boys.

Examiner

PKH

QUESTION ONE Start a new page.

(a) Solve the quadratic equation $x^2 = 9$.

(b) What is the radius of the circle $x^2 + y^2 = 16$?

(c) Find the gradient of the line $2y = 6x - 4$.

(d) Find the volume of a cube with edge length 5 cm.

(e) Factorise $3x^2 - x$.

(f) Find the simple interest earned on \$2000 for 3 years at 6% per annum.

(g) Expand and simplify $(3x - 2)(2x + 1)$.

(h) Seven cards are labelled 1, 2, 3, 4, 5, 6 and 7. One card is chosen at random. What is the probability that the card has an odd number on it?

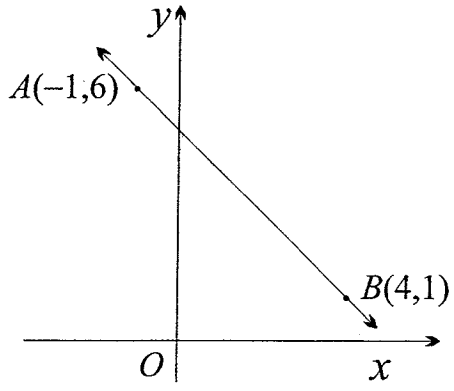
(i) Sketch the line $y = 2x + 3$.

(j) Find the y-intercept of the parabola $y = x^2 + x - 7$.

QUESTION TWO Start a new page.

(a) Write down the equation of the line with gradient -2 and y-intercept 6.

(b)



(i) Find the length of the interval AB.

(ii) Find the gradient of the line passing through the points A and B.

(iii) Find the equation of the line passing through the points A and B. Give your answer in general form.

Exam continues next page ...

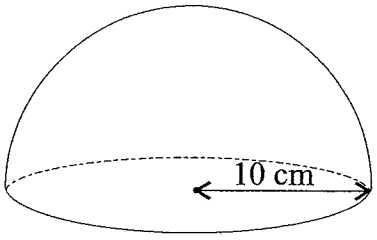
(c) (i) Solve $x^2 - 2x = 0$ by factorising.

(ii) Solve $x^2 + 4x - 2 = 0$ by using the quadratic formula. (Leave your answer in exact form.)

(iii) Solve $x^2 + 6x - 4 = 0$ by completing the square. (Leave your answer in exact form.)

QUESTION THREE Start a new page.

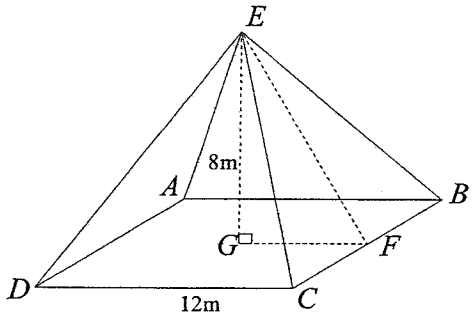
(a)



(i) Find the volume of the hemisphere, with radius of 10cm, drawn above. (Give your answer correct to two decimal places.)

(ii) Find the total surface area of the hemisphere, including the circular base. (Give your answer correct to two decimal places.)

(b)



The diagram above represents a monument which is built in the shape of a square pyramid with the edge of the base being 12 metres and vertical height being 8 metres.

(i) (α) Find the volume of the pyramid.

(β) The monument is built out of concrete and concrete costs \$223 per cubic metre. Find the total cost of the concrete.

(ii) (α) Find the length of the slant height EF marked on the diagram.

(β) The triangular faces of the pyramid, which are above ground, are to have a protective coating which costs \$80 per square metre. How much will the protective coating cost?

Exam continues overleaf ...

QUESTION FOUR Start a new page.

(a) Consider the parabola $y = x^2 - 2x - 8$.

(i) Find the x-intercepts.

(ii) Find the axis of symmetry.

(iii) Find the coordinates of the vertex.

(iv) Draw a neat sketch of the parabola, showing the features above.

(b) Draw neat graphs of the following on separate axes. Show all significant features. (Use about a third of a page for each.)

(i) $y = 3^x$

(ii) $y = -\dfrac{4}{x}$

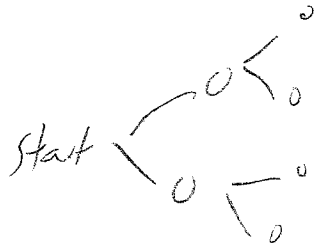
(c) Two different cards are chosen at random from 26 cards each marked with a different letter of the alphabet. The vowels are A, E, I, O and U.

(i) Find the probability that both cards are vowels.

(ii) Find the probability that neither of the cards is a vowel.

(iii) Find the probability that at least one card is a vowel.

$\frac{92}{325}$

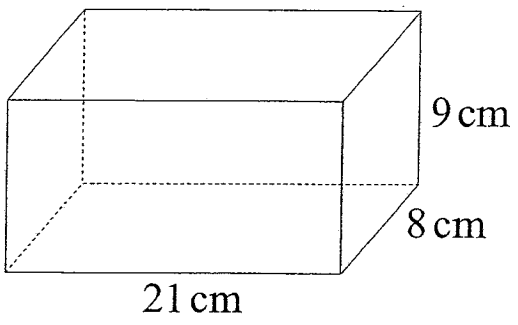


Exam continues next page ...

QUESTION FIVE Start a new page.

(a) A new car is purchased for \$55 000 and it depreciates at a rate of 15% per annum. Find its value after three years, correct to the nearest hundred dollars.

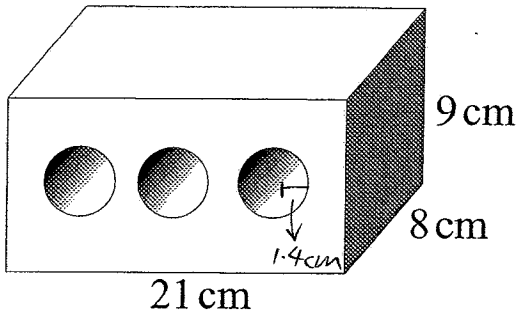
(b)



A clay brick in the shape of a rectangular prism has dimensions as shown in the diagram above.

(i) Find the volume of the clay brick.

(ii) Three identical holes are drilled through the brick, as shown in the diagram below. Each hole has radius 1·4 cm.



(α) What is the volume of clay remaining in the brick after the holes have been made? (Give your answer correct to the nearest cubic centimetre.)

(β) What percentage of clay is removed by making these three holes through the brick? (Give your answer correct to one decimal place.)

(c) Find the centre and radius of the circle $x^2 + 4x + y^2 - 6y = 0$.

$(-2, 3)$ $\sqrt{13}$

Exam continues overleaf ...

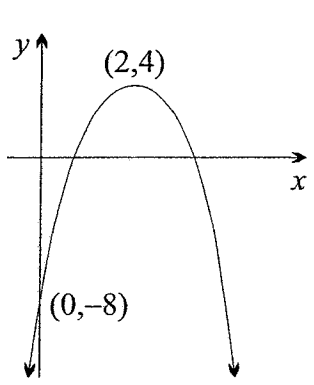
QUESTION SIX Start a new page.

(a) A multiple choice Science paper has 4 possible answers for each question, only one of which is correct. Edgar guesses the last five questions. What is the probability that he gets at least one correct?

(b) Solve the equation $\dfrac{2x}{x+1} + \dfrac{1}{x+2} = 1$. (Leave your answer in exact form.)

(c) A person invests \$8000 at $r\%$ per annum compound interest over 4 years. The investment grows to \$10 872. Find the value of r correct to one decimal place if the interest is compounded annually.

(d)



Find the equation of the parabola sketched above which has vertex at $(2, 4)$ and y-intercept at $(0, -8)$.

Exam continues next page ...

QUESTION SEVEN Start a new page.

(a) (i) Find the midpoint of the interval joining $A(-1,3)$ and $B(-3,9)$.

(ii) Show that the equation of the perpendicular bisector of the interval joining $A(-1,3)$ and $B(-3,9)$ is $x-3y+20=0$.

(iii) The perpendicular bisector in part (ii) cuts the y-axis at M and the x-axis at N. Find the area of the triangle MON, where O is the origin.

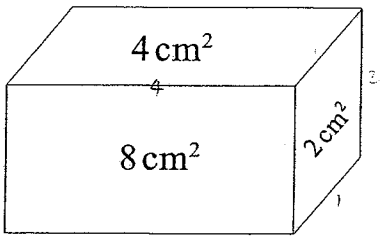
(b) Solve the equation $x+\dfrac{4x}{x^2-5}=0$.

(c) (i) Copy and complete the table of values for $y=4-2^{-x}$. (Leave your answers correct to two decimal places.)

x	-3	-2	-1	0	1	2	3
y							

(ii) Sketch $y=4-2^{-x}$.
(Your sketch should take up about a third of a page.)

(d)



The diagram above represents a box in the shape of a rectangular prism. The areas of the three adjacent faces shown are $2\,\text{cm}^2$, $4\,\text{cm}^2$ and $8\,\text{cm}^2$. Find the volume of the box.

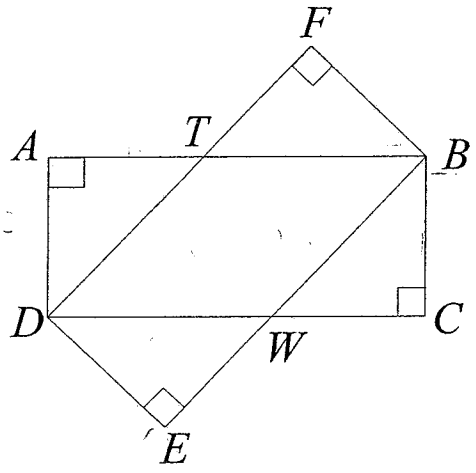
Exam continues overleaf ...

QUESTION EIGHT Start a new page.

(a) (i) Explain why $x^8 - 1 = 0$ has only two real solutions.

(ii) Find all real solutions to the equation $x^{10} - 2 = 2x^8 - x^2$.

(b)



In the diagram above, $ABCD$ and $DEBF$ are two congruent rectangles with sides 3 units and 7 units.

(i) Show that $AT = \dfrac{20}{7}$ units.

(ii) A pyramid with $DTBW$ as base has volume 58 cubic units. Find the height of the pyramid.

(c) A piece of wire 100 cm long is cut and the two pieces are bent to form a square and a circle. The piece bent to form a square is x cm long.

The total area of the square and the circle is A square centimetres.

(i) (α) Show that $A = \left(\dfrac{\pi + 4}{16\pi}\right) x^2 - \dfrac{50}{\pi}x + \dfrac{2500}{\pi}$.

(β) Find the value of x for which the area A is minimised. (Write your answer in simplest form in terms of π.)

(ii) How large can A be?

END OF EXAMINATION

2006 SOLUTION TO FORM IV HALF YEARLY (SGS)

Question 1 (13 marks)

(a) $x^2 = 9$
$x = \pm 3$ ✓

(b) $x^2 + y^2 = 16$
radius $= 4$ ✓

(c) $2y = 6x - 4$
$y = 3x - 2$
gradient $= 3$ ✓

(d) $V = 5^3 = 125\ cm^3$ ✓

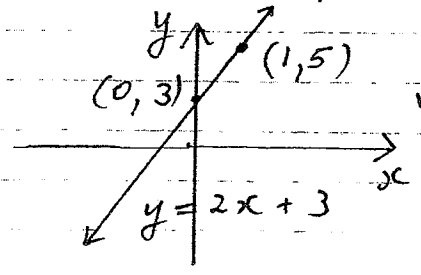
(e) $3x^2 - x$
$= x(3x - 1)$ ✓

(f) $S.I = P \times r \times n$
$= 2000 \times \frac{6}{100} \times 3$ ✓
$= \$360$ ✓

(g) $(3x-2)(2x+1)$
$= 6x^2 + 3x - 4x - 2$ ✓
$= 6x^2 - x - 2$ ✓

(h) $P(odd) = \frac{4}{7}$ ✓

(i)



✓ ✓

(j) y-int when $x = 0$
y-int $= -7$ ✓

Question 2 (13 marks)

(a) Eqn is $y = mx + b$ ✓
$y = -2x + 6$

(b) (i) $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$= \sqrt{(-1-4)^2 + (6-1)^2}$ ✓
$= \sqrt{25 + 25}$
$= \sqrt{50}$ ✓
$= 5\sqrt{2}$

(ii) gradient $= \frac{y_2 - y_1}{x_2 - x_1}$
$= \frac{6-1}{-4-1}$ ✓
$= -1$ ✓

(iii) Eqn of the line is
$y - y_1 = -1(x - x_1)$
$y - 1 = -1($ ✓
$y - 1 = -x + 4$
$x + y - 5 = 0$ ✓

(c) (i) $x^2 - 2x = 0$
$x(x-2) = 0$ ✓
$x = 0$ or $x = 2$ ✓

(ii) $x^2 + 4x - 2 = 0$
$x = \frac{-4 \pm \sqrt{16+8}}{2}$ ✓
$x = \frac{-4 \pm \sqrt{24}}{2}$ ✓
$x = -2 \pm \sqrt{6}$

(iii) $x^2 + 6x - 4 = 0$
$x^2 + 6x + 9 = 4 + 9$
$(x+3)^2 = 13$ ✓
$x = -3 \pm \sqrt{13}$ ✓

Question 3 (13 marks)

(a) (i) $V = \frac{2}{3}\pi r^3$

$= \frac{2}{3}\pi 10^3$ ✓

$= 2094.40 cm^3$ ✓

(ii) $SA = 3\pi r^2$ ✓

$= 3 \times \pi \times 10^2$ ✓

$= 942.48 cm^2$ ✓

(b) (i) (α) $V = \frac{1}{3} A h$

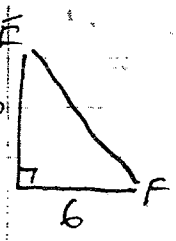
$= \frac{1}{3} \times 12^2 \times 8$ ✓

$= 384 m^3$ ✓

(β) Cost $= 384 \times 223$ ✓

$= \$85,632$ ✓

(ii) (α)



$EF^2 = 6^2 + 8^2$ ✓

$EF = 10$ ✓

(β) SA (of Δ faces) $= 4 \times \frac{1}{2} \times b \times h$

$= 2 \times 12 \times 10$ ✓

$= 240 m^2$ ✓

Cost $= 240 \times 80$

$= \$19200$ ✓

Question 4 (13 marks)

(a) (i) $y = x^2 - 2x - 8$

x-int when $y = 0$

$x^2 - 2x - 8 = 0$ ✓

$(x-4)(x+2) = 0$

$x = 4$ or $x = -2$ ✓

(ii) axis of symmetry is

$x = \frac{4 + -2}{2}$ ✓

ie $x = 1$

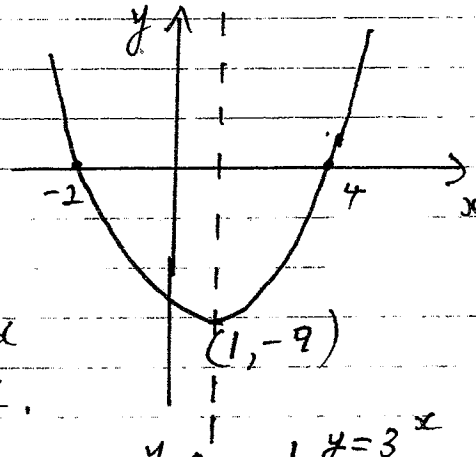
(iii) when $x = 1$

$y = 1 - 2 - 8$

$y = -9$ ✓

co-ordinates of vertex is $(1, -9)$

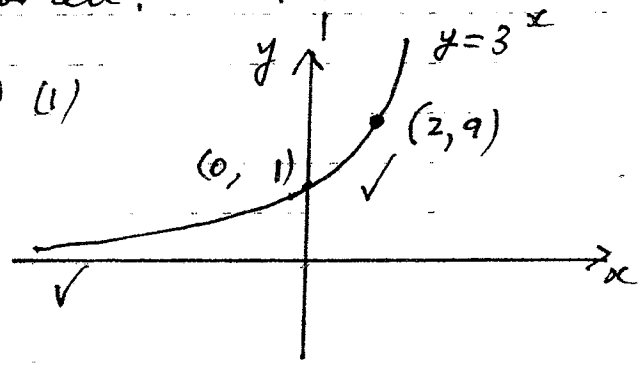
(iv)



✓ general shape

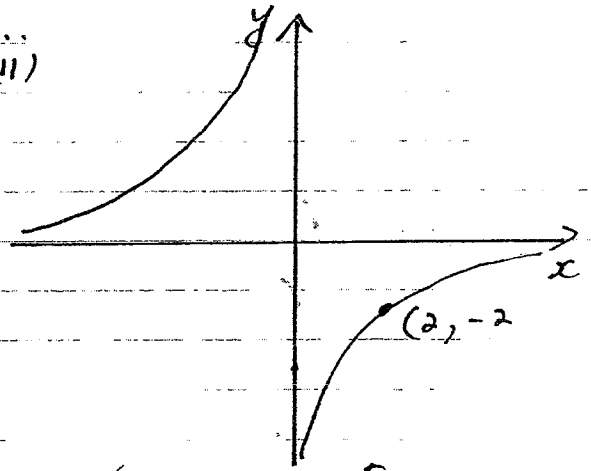
✓ roots and vertex marked.

(b) (i)



✓ ✓

(ii)



✓ quadrants

✓ asymptotic behaviour

(b) (i) $P(\text{both vowels}) = \frac{5}{26} \times \frac{4}{25}$ ✓

$= \frac{2}{65}$

(ii) $P(\text{neither is a vowel}) = \frac{21}{26} \times \frac{20}{25}$ ✓

$= \frac{42}{65}$

(iii) $P(\text{at least one vowel}) = 1 - P(\text{no vowels})$

(complementary events) $= 1 - \frac{42}{65} = \frac{23}{65}$ ✓

Question 5 (13 marks)

(a) $V_n = V\left(1-\frac{r}{100}\right)^n$ ✓

$V_3 = 55000\left(1-\frac{15}{100}\right)^3$ ✓

$= 55000 \times (0.85)^3$

$= \$33,800$ (nearest hundred dollars) ✓

(b) (i) Volume $= 21 \times 9 \times 8$

$= 1512 \text{ cm}^3$ ✓

(ii) Volume of cylinder

$= \pi r^2 h$

$= \pi (1.4)^2 \times 8$

$= 49.2601$ ✓

Volume of 3 cylinders

$= 3 \times 49.2601$ ✓

$= 148 \text{ cm}^3$

Volume remaining

$= 1512 - 148$ ✓

$= 1364 \text{ cm}^3$

Percentage removed

$= \frac{148}{1512} \times \frac{100}{1}$ ✓

$= 9.8\%$ ✓

(c) $x^2 + 4x + y^2 - 6y = 0$

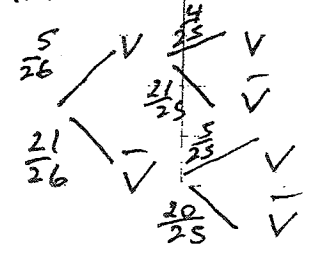
$x^2 + 4x + 4 + y^2 - 6y + 9 = 13$ ✓

$(x+2)^2 + (y-3)^2 = 13$ ✓

Centre $= (-2, 3)$ ✓

Radius $= \sqrt{13}$ units ✓

TREE DIAGRAM FOR Q4(b)



V = vowel

V̄ = not vowel

Question 6 (13 marks)

TREE DIAGRAM

$\frac{1}{4}$ C / $\bar{C}$ etc

$\frac{3}{4}$ $\bar{C}$ — $\bar{C}$ — $\bar{C}$ ✓ ($\frac{3}{4}$, $\frac{3}{4}$) (complementary events)

P(all wrong) $= \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4}$

(a) P(he gets at least one correct)

$= 1 - P$(he gets them all wrong) ✓

$= 1 - \left(\frac{3}{4}\right)^5$ ✓

$= 1 - \frac{243}{1024}$

$= \frac{781}{1024}$ (or equivalent decimal) ✓

(b) $\frac{2x}{x+1} + \frac{1}{x+2} = 1$

$\times (x+2)(x+1)$:

$2x(x+2) + x + 1 = (x+1)(x+2)$ ✓

$2x^2 + 4x + x + 1 = x^2 + 3x + 2$ ✓

$x^2 + 2x - 1 = 0$ ✓

$x = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}$ ✓

(c) $P_n = P\left(1 + \frac{r}{100}\right)^n$

$10872 = 8000 \times \left(1 + \frac{r}{100}\right)^4$ ✓

$1 + \frac{r}{100} = \sqrt[4]{\frac{10872}{8000}}$ ✓

$1 + \frac{r}{100} \doteq 1.0797$ ✓

Interest rate is 8.0% per annum.

(d) $y = a(x-2)^2 + 4$ since vertex is 2 units across and 4 units up

substitute the y int — $(0, -8)$ ✓

$-8 = 4a + 4$

$4a = -12$ ✓

$a = -3$

Equation is $y = -3(x-2)^2 + 4$ ✓

$y = -3(x^2 - 4x + 4) + 4$

$y = -3x^2 + 12x - 8$

Question 7 (13 marks)

(a) (i) Midpoint $= \left(\frac{-1+-3}{2}, \frac{3+9}{2}\right)$ ✓

$= (-2, 6)$

(ii) grad of AB $= \frac{9-3}{-3--1} = -3$

grad of $\perp = \frac{1}{3}$ ✓

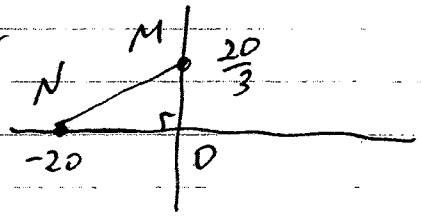
Eqn of $\perp$ bisector is $y - 6 = \frac{1}{3}(x+2)$

$3y - 18 = x + 2$ ✓

$x - 3y + 20 = 0$

(iii) When $x = 0$, $y = \frac{20}{3}$ ✓

$y = 0$, $x = -20$



Area $= \frac{20}{3} \times 20 \times \frac{1}{2}$ ✓

$= \frac{200}{3}$ units^2 ✓

(b) $x + \frac{4x}{x^2-5} = 0$ — *

$x\left(1 + \frac{4}{x^2-5}\right) = 0$ ✓

$x = 0$ or $1 + \frac{4}{x^2-5} = 0$ ✓

$x = 0$ or $x^2 - 5 \quad 4 = 0$

$x = 0$ or $x^2 = 1$ ✓

$x = 0, 1$ or -1

or $x(x^2-5) + 4x = 0$ ✓

$x^3 - 5x + 4x = 0$

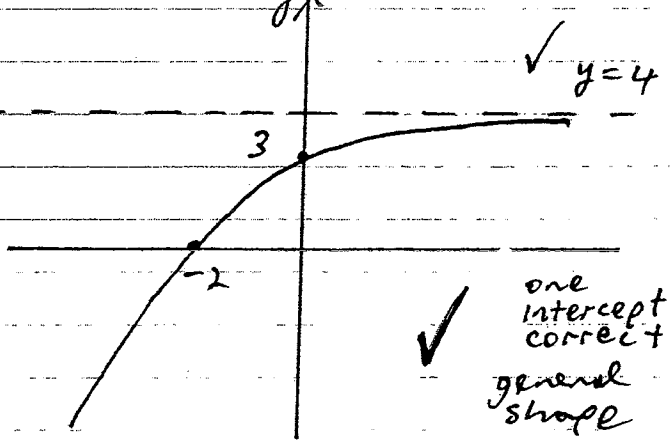
$x^3 - x = 0$ ✓

$x(x^2-1) = 0$

$x = 0, 1$ or -1 ✓

(c) $y = 4 - 2^{-x}$ ✓ at least 5 correct

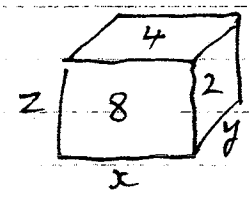
x	-3	-2	-1	0	1	2	3
y	-4	0	2	3	$3\frac{1}{2}$	$3\frac{3}{4}$	3



✓ one intercept correct

✓ general shape

(d)



Let sides be x, y and z

$xy \times xz \times yz$

$= 4 \times 2 \times 8$ ✓

$\therefore x^2y^2z^2 = 64$

$xyz = 8$ ✓

Volume $= 8$ cm^3

Question 8 (13 marks)

(a) (i) $x^8 - 1 = 0$

$(x^4-1)(x^4+1) = 0$

$(x^2-1)(x^2+1)(x^4+1) = 0$

$x^2 = 1$, $x^2 = -1$, $x^4 = -1$ ✓

(even powers of a real number cannot be negative)

So $x = \pm 1$ are the only real solutions. ✓

(ii) $x^{10} - 2 = -x^2 + 2x^8$

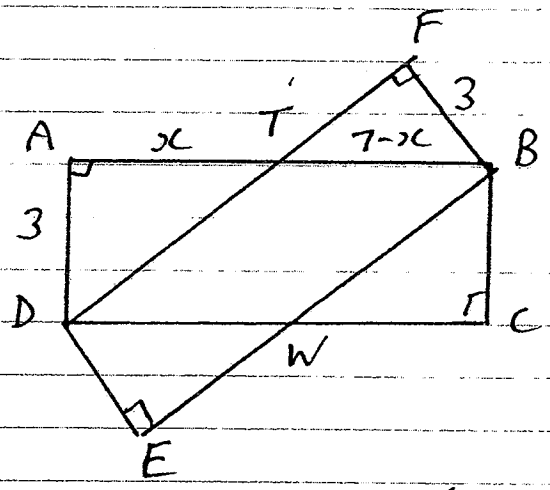
$x^{10} - 2x^8 + x^2 - 2 = 0$

$x^8(x^2-2) + 1(x^2-2) = 0$

$(x^2-2)(x^8+1) = 0$

Solutions are: $x = \pm\sqrt{2}$ ✓

(b)



Let $AT = x$

$TB = 7 - x$

$\Delta ADT \equiv \Delta BFT$ (A.A.S) ✓ (Reasons not necessary)

$\therefore DT = 7 - x$

By Pythagoras $3^2 + x^2 = (7-x)^2$ ✓

$x^2 + 9 = 49 - 14x + x^2$

$14x = 40$

$x = \frac{20}{7}$ ✓

Area of $\parallel$ TBWD $= b \times h$

$= 3\left(7 - \frac{20}{7}\right) = \frac{87}{7}$ units^2

Let A = area of base, h = height of pyramid

Volume of pyramid $= \frac{1}{3} A h$ ✓

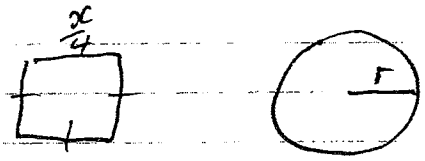
$58 = \frac{1}{3} \times \frac{87}{7} \times h$

$\div 29$: $2 = \frac{1}{3} \times \frac{3}{7}$ So $h = 14$ units. ✓

Question 8 (c)

(i) (α)

x	$100-x$



$2\pi r = 100 - x$

$r = \frac{100-x}{2\pi}$

$A = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{100-x}{2\pi}\right)^2$ ✓

$= \frac{x^2}{16} + \frac{10000 - 200x + x^2}{4\pi}$ ✓

$= x^2\left(\frac{1}{16} + \frac{1}{4\pi}\right) - \frac{50}{\pi}x + \frac{2500}{\pi}$

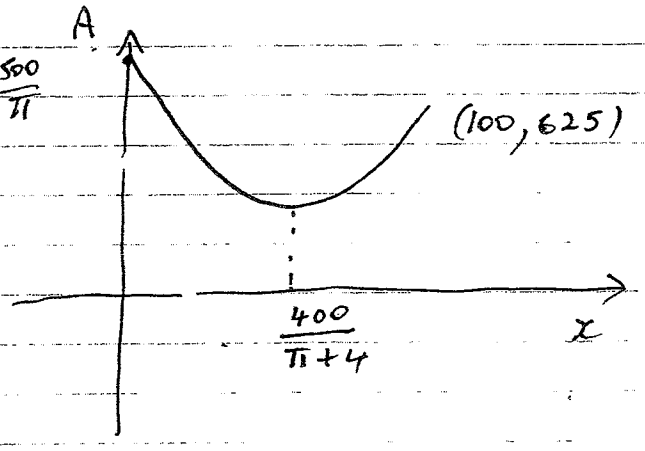
(β) Minimum occurs at vertex of the parabola

∴ So $x = \frac{-b}{2a}$

$x = \frac{\frac{50}{\pi}}{\frac{1}{8} + \frac{1}{2\pi}}$ ✓

$= \frac{400}{\pi + 4}$

(ii)



$x = 0$

$A = \frac{2500}{\pi} \doteqdot 795.77$

$x = 100$

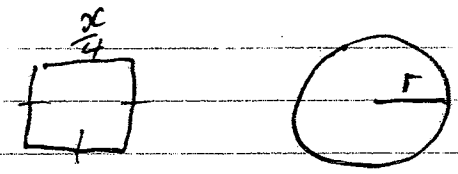
$A = (25)^2$

$= 625$ ✓

Clearly from graph the longest A can be is $\frac{2500}{\pi}$ units ✓

Question 8 (c)

(i) (α) $\underline{x \quad | \quad 100-x}$



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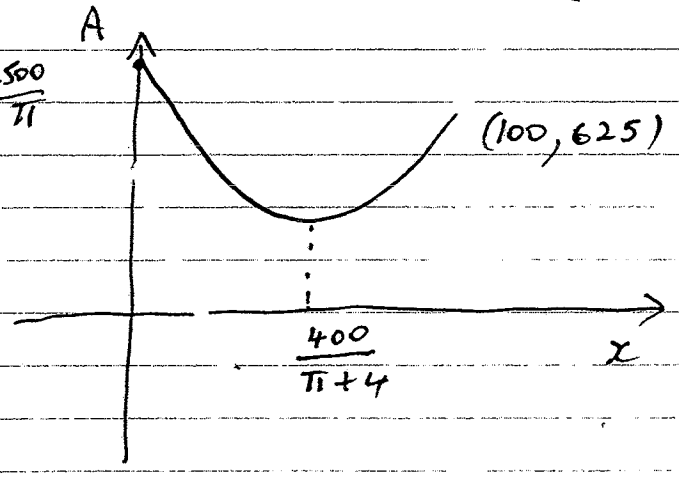

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