SYDNEY GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT
YEARLY EXAMINATIONS 2006

FORM IV

MATHEMATICS

Examination date

Thursday 2nd November 2006

Time allowed

Two hours

Instructions

All eight questions may be attempted.
All eight questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.

Collection

Write your name, class and master clearly on the front.
Hand in all the writing paper in a single well-stapled bundle.
Keep the printed examination paper and bring it to your next Mathematics lesson.

4A: PKH	4B: LYL	4C: JCM	4D: GJ
4E: JMR	4F: REP	4G: BDD	4H: SJE
4I: DNW	4J: KWM		

Checklist

Writing paper required.
Candidature: 192 boys.

Examiner

PKH

QUESTION ONE Start a new page.

(a) Simplify $-3x + 7 + 7x$.

(b) Solve the equation $(x + 1)(x - 2) = 0$.

(c) What is the radius of the circle $x^2 + y^2 = 16$?

(d) Find the gradient of the line $y = -2x + 3$.

(e) Find the value of $\sin 37^\circ$ correct to two decimal places.

(f) Find the value of $x^2 + 4x$ when $x = -4$.

(g) Write down the exact value of $\tan 60^\circ$.

(h) Factorise $x^2 - 2x - 8$.

(i) Expand and simpify $(a - 3)(a + 3)$.

(j) Find the gradient of the line through $A(0, 6)$ and $B(3, 0)$.

(k) What monthly interest rate is equivalent to 9% per annum?

(l) Solve the inequation $-2x \leq 6$ and graph your solution on a number line.

(m) Write without a fractional index $x^{-\frac{1}{2}}$.

Exam continues next page ...

QUESTION TWO Start a new page.

(a) Solve the equation $3x - 5 = 3 - 2x$.

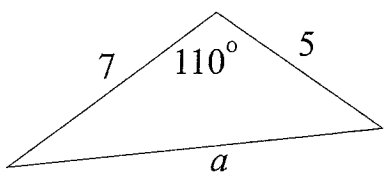
(b) Find the compound interest earned on \$10 000 at 5% per annum for 4 years compounded annually.

(c) Find the exact value of $\cos 210°$.

(d) Solve the equation $\sin\theta = -\frac{1}{\sqrt{2}}$ for $0° \leq \theta \leq 360°$.

(e) Simplify $\left(2\sqrt{5}\right)^2$.

(f)



In the diagram above, use the cosine rule to find a correct to two decimal places.

(g) Two similar solids have edge lengths in the ratio 1 : 2.

(i) Write down the ratio of the volumes.

(ii) If the smaller solid has volume of 12 units^3 find the volume of the larger solid.

Exam continues overleaf ...

QUESTION THREE Start a new page.

(a) On separate number planes, sketch the following showing the important features. Use about a third of a page for each graph.

(i) $y = -2x + 3$

(ii) $y = \sin x$ for $0° \leq x \leq 360°$.

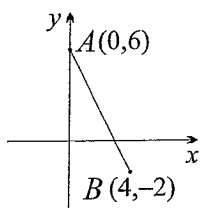
(b) Sketch the parabola $y = 4x - x^2$ showing the x-intercepts and the vertex.

(c) Simplify the following:

(i) $\log_7 7$

(ii) $\log_{10} 12 - \log_{10} 4$

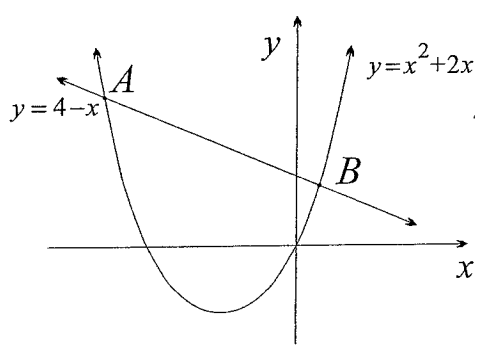
(d)



(i) Find the midpoint of the interval joining points A and B.

(ii) Find the equation of the line AB. Leave your answer in the form $y = mx + b$.

(e)



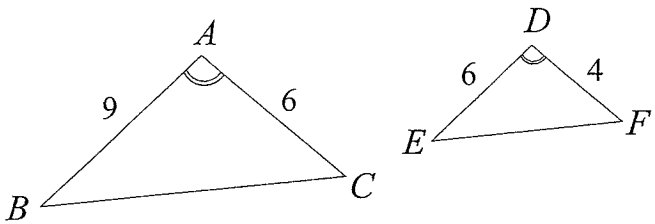
On the diagram above the parabola $y = x^2 + 2x$ and the line $y = 4 - x$ are drawn.
Their points of intersection are represented by A and B.
Solve a pair of simultaneous equations to find the coordinates of points A and B.

Exam continues next page ...

QUESTION FOUR Start a new page.

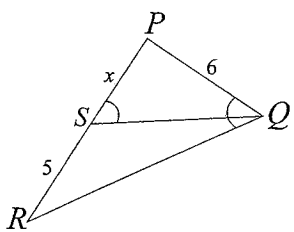
(a) On the same set of axes sketch the graphs of $y = 3^x$, $y = x$ and $y = \log_3 x$. Use about one third of a page.

(b)



In the triangles above $\angle A = \angle D$.
Prove that the triangles are similar.

(c)



In the diagram above $PQ = 6$ units, $RS = 5$ units and $PS = x$ units.
$\triangle PQR \,|||\, \triangle PSQ$ by the AA test.

(i) (α) Explain why $\dfrac{x+5}{6} = \dfrac{6}{x}$.

(β) Hence find the value of x.

(ii) If the area of $\triangle PSQ$ is k square units, find the area of $\triangle RSQ$ in terms of k.

(d) In triangle ABC, $\angle B = 42°$, $c = 7$ and $b = 5$. Find the possible sizes of $\angle C$ correct to the nearest degree.

(e) Simplify $\dfrac{t-1}{1-\dfrac{1}{t}}$.

Exam continues overleaf ...

QUESTION FIVE Start a new page.

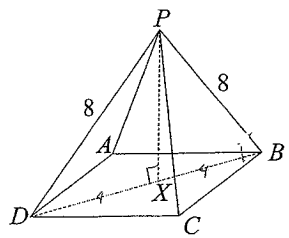
(a) Factorise the following:

(i) $x^2 - 64$

(ii) $x^3 - 64$

(b) Find a pair of integers p and n so that $2\log_6 12 + \log_6 3 - \log_6 18 = p + \log_6 n$.

(c)



The diagram above shows a pyramid with square base $ABCD$. Point P is the apex of the pyramid. It is given that $PD = PB = 8$ and $\angle PBD = 60°$. The point P lies vertically above the centre X of the square.

(i) Find length DB giving reasons.

(ii) Find the exact volume of the pyramid.

(d) Given that $\sin\theta = \dfrac{2}{3}$ and θ is obtuse, find the exact value of $\tan\theta$.

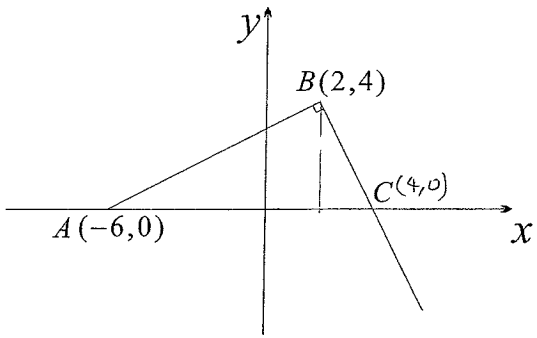
(e) By completing the square find the minimum value of the expression $x^2 + 4x + 7$.

Exam continues next page ...

QUESTION SIX Start a new page.

(a) Solve the equation $\cos 2\theta = 0$, for $0° \leq \theta \leq 360°$.

(b)



The diagram above shows the points $A(-6,0)$ and $B(2,4)$. The line BC is perpendicular to the line AB and point C lies on the x-axis.

(i) Find the gradient of the line AB.

(ii) Write down the gradient of the line BC.

(iii) Show that the equation of the line BC is $2x + y - 8 = 0$.

(iv) Find the coordinates of point C.

(v) Find the area of triangle ABC.

(vi) Find the equation of the circle with diameter AC.

(vii) Find the volume of a hemisphere, whose base is a circle with AC as diameter. Leave ~~your~~ answer in exact form.

(c) Simplify $7^{2\log_7 a}$.

Exam continues overleaf ...

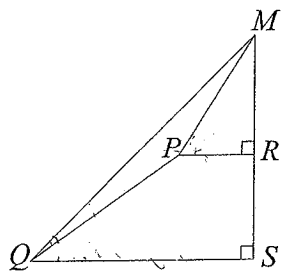
QUESTION SEVEN Start a new page.

(a) Solve the equation $3^{2-x} = 9^x$.

(b) Solve the following pair of simultaneous equations:

$$x - y = 10$$
$$\frac{12}{y} - \frac{12}{x} = 5$$

(c)



In the diagram above, P and Q are two observation points lying in the the same vertical plane as M, the summit of a mountain. The following observations are made: $\angle RPM = 42°, \angle SQM = 33°$ and $\angle SQP = 14°$. From contour lines on a map it is known that Q is 200 metres above sea level and P is 600 metres above sea level.

(i) Copy the diagram and find the size of $\angle QMP$.

(ii) Find the length of PQ correct to two decimal places.

(iii) Hence, or otherwise, find the height of the mountain, above sea level, to the nearest ten metres.

Exam continues next page ...

QUESTION EIGHT Start a new page.

(a) Simplify the following expression and write your answer as a surd:

$$\left(\sqrt[3]{4} \times \frac{1}{\sqrt[6]{8}} \times \sqrt[12]{2^{-1}}\right)^4.$$

(b) Show that the equation $\log_{10}\left(2^x + \frac{475}{2^x}\right) = 2$ is satisfied by two values of x whose sum is $\frac{\log_{10} 475}{\log_{10} 2}$.

(c) The Fermat numbers are defined by the rule that $F_n = 2^{2^n} + 1$. The first four Fermat numbers are $F_1 = 5, F_2 = 17, F_3 = 257$ and $F_4 = 655\,537$. They are all prime and Fermat thought that $F_5 = 2^{2^5} + 1 = 4\,294\,967\,297$ was also prime.

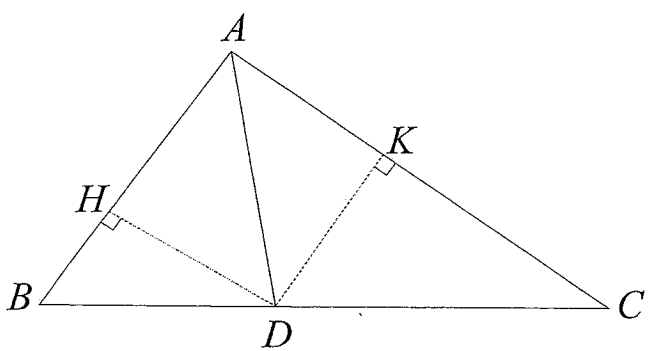
It is known that $641 = 2^4 + 5^4 = 5 \times 2^7 + 1$.

Without using a calculator:

(i) show that 641 is a factor of both $5^4 \times 2^{28} + 2^{32}$ and $5^4 \times 2^{28} - 1$,

(ii) show that F_5 is not prime.

(d)



In the diagram above, triangle ABC is acute angled and $\frac{AB}{BD} = \frac{AC}{CD}$. The lines DH and DK are respective perpendiculars to AB and AC.

(i) Explain why $\frac{\text{area of } \triangle ABD}{\text{area of } \triangle ACD} = \frac{BD}{DC}$.

(ii) Hence prove that the line DA bisects $\angle BAC$.

(iii) Prove the converse of the result. That is, prove that if DA bisects $\angle BAC$ then $\frac{AB}{BD} = \frac{AC}{CD}$.

END OF EXAMINATION

FORM IV Solutions to SGS Maths Yearly 2006

1 (a) $-3x + 7 + 7x$

$= 4x + 7$ ✓

(b) $(x+1)(x-2) = 0$ ✓

$x = -1$ or $x = 2$

(c) radius = 4 units ✓

(d) gradient = -2 ✓

(e) $\sin 37^\circ \doteq 0.60$ ✓ (must be correct to two places)

(f) $x^2 + 4x = (-4)^2 + 4 \times -4 = 16 - 16 = 0$ ✓

(g) $\tan 60^\circ = \sqrt{3}$ ✓

(h) $x^2 - 2x - 8 = (x-4)(x+2)$ ✓

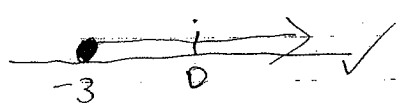
(i) $(a-3)(a+3) = a^2 - 9$ ✓

(j) gradient = $\frac{6-0}{0-3} = -2$ ✓

(k) 9% p.a $= \frac{9}{12} = 0.75\%$ per month ✓

(l) $-2x \leq 6$ ✓

$x \geqslant -3$



✓

(m) $x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$ ✓

14

Question 2

(a) $3x - 5 = 3 - 2x$ ✓

$5x = 8$ ✓

$x = \frac{8}{5}$ ✓

(b) $P_4 = 1000 \times (1.05)^4$ ✓

$= \$12\,155.06$

Interest $= \$2155.06$ ✓ Ignore error in cents

(c) $\cos 210^\circ = -\cos 30^\circ$ ✓

$= -\frac{\sqrt{3}}{2}$ ✓

(d) $\sin\theta = -\frac{1}{\sqrt{2}}$ Related angle of 45°

$\theta = 180^\circ + 45^\circ$ or $360^\circ - 45^\circ$ ✓

$= 225^\circ$ or 315°

✓ ✓

(e) $(2\sqrt{5})^2 = 4 \times 5 = 20$ ✓

(f) $a^2 = 7^2 + 5^2 - 2 \times 7 \times 5 \cos 110^\circ$ ✓

$a = 9.90$ ✓ (Dont penalise rounding error)

(g) (i) Ratio of volumes is 1 : 8 ✓

(ii) Larger volume $= 8 \times 12$

$= 96$ units^3 ✓

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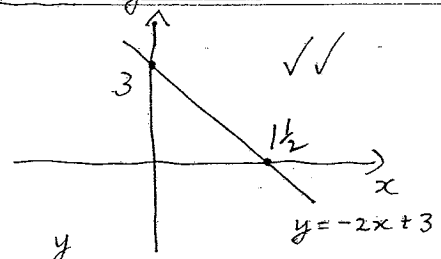
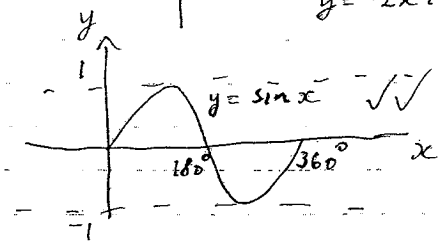
Question 3

(a) (i)



✓✓

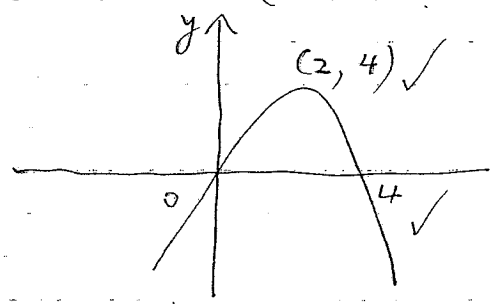
(ii)



✓✓

(b) $y = 4x - x^2$

$= x(4 - x)$



✓ ✓ ✓

(c) Points of intersection where

$x^2 + 2x = 4 - x$ ✓

$x^2 + 3x - 4 = 0$

$(x + 4)(x - 1) = 0$

$x = -4$ and $x = 1$ ✓

$y = 8$ $y = 3$

So required points are

$(-4, 8)$ and $(1, 3)$ ✓

✓

(c) (i) $\log_7 7 = 1$

(ii) $\log_{10} 12 - \log_{10} 4$

$= \log_{10} 3$ ✓

(d) (i) $M = (2, 2)$ ✓

(ii) gradient $= \frac{6 - -2}{0 - 4}$

$= -2$ ✓

Equ of line is

$y = mx + b$

$y = -2x + b$ ✓

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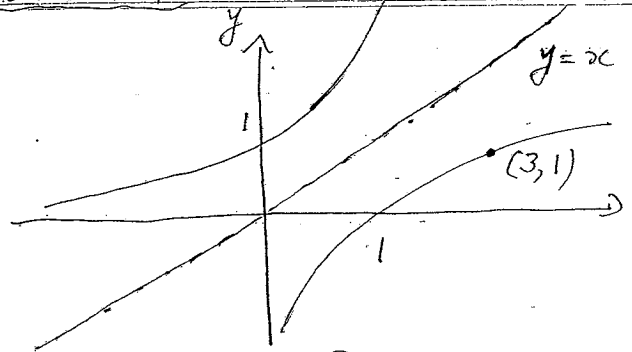
Question 4

(a)



Shapes will do.

(b) $\angle A = \angle D$, $\frac{9}{6} = \frac{6}{4}$ ✓

$\Delta ABC \,|||\, \Delta DEF$ (sides surrounding equal angles are in proportion) ✓

(c) (α) $\frac{x+5}{6} = \frac{6}{x}$ (sides of similar Δ's are in the same ratio) ✓

(β) $x^2 + 5x = 36$

$x^2 + 5x - 36 = 0$ ✓

$(x+9)(x-4) = 0$

$x = -9$ or $x = 4$ ✓

So $x = 4$

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(ii) Ratio of lengths $= \frac{9}{6} = \frac{3}{2}$ ✓

So the area of $\Delta PRQ = K \times \frac{9}{4}$

Area of $\Delta SRQ = \frac{9}{4}K - K = \frac{5K}{4}$ ✓

(d) $\frac{\sin C}{c} = \frac{\sin B}{b}$ ✓

$\sin C = \frac{7 \sin 42^\circ}{5}$ ✓

$C \doteq 70^\circ$ or 110°

Both are valid. ✓

(e) $\frac{t-1}{1-\frac{1}{t}}$

$= \frac{t^2 - t}{t-1} = \frac{t(t-1)}{t-1}$

$= t$ ✓

Question 5

(a) (i) $x^2 - 64 = (x-8)(x+8)$ ✓

(ii) $x^3 - 64 = (x-4)(x^2+4x+16)$ ✓

(b) $2\log_6 12 + \log_6 3 - \log_6 18$

$= \log_6 \frac{144 \times 3}{18}$ ✓

$= \log_6 24$

$= \log_6 4 + \log_6 6$ ✓

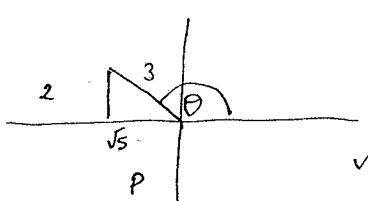
$= \log_6 4 + 1$ ✓

So $n = 4$ and $p = 1$

(c) (i) ΔPDB is equilateral since it is an isosceles Δ with a 60° angle.

$\therefore BD = 8$. ✓

(d)



$\tan\theta = \frac{-2}{\sqrt{5}}$ ✓

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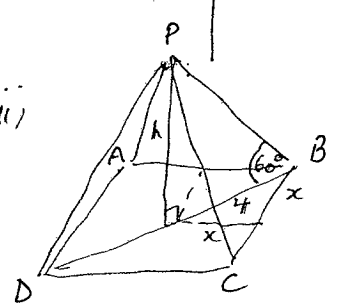
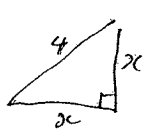
(ii)





$x^2 + x^2 = 16$

$x^2 = 8$

$x = \sqrt{8}$ ✓

Area of base $= (2\sqrt{8})^2 = 32$. ✓

$\frac{h}{4} = \tan 60^\circ = \sqrt{3}$ ✓

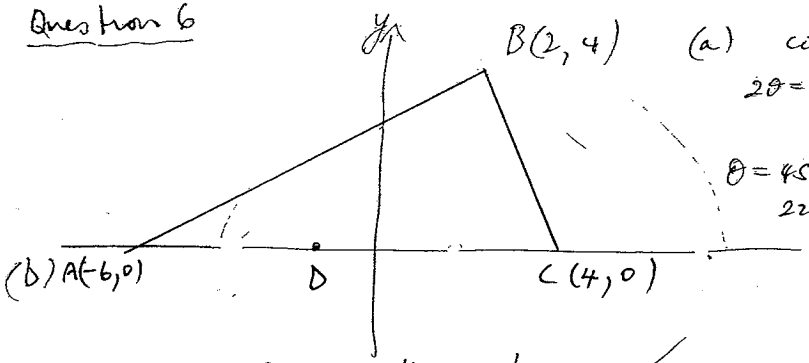
$h = 4\sqrt{3}$

Volume $= \frac{1}{3}Ah = \frac{1}{3} \times 32 \times 4\sqrt{3}$ ✓

$= \frac{128\sqrt{3}}{3}$ units^3

(e) $x^2 + 4x + 7$ ✓ least value

Question 6



(a) $\cos 2\theta = 0$

$2\theta = 90^\circ, 270^\circ$

$450^\circ, 630$

$\theta = 45^\circ, 135^\circ$ ✓

$225^\circ, 315^\circ$ ✓

(i) $m(AB) = \frac{4}{2+6} = \frac{1}{2}$ ✓

(ii) $m(BC) = -2$ ✓

(iii) Eqn of BC is ✓

$y - 4 = -2(x-2)$ ✓

$2x + y - 8 = 0$

(iv) $C = (4, 0)$ ✓

(v) Area $\Delta ABC = \frac{1}{2} \times 4 \times 10$ ✓

$= 20$ ✓

(vi) Let D be the centre ✓

$D = (1, 0)$

Eqn of circle is

$(x-1)^2 + y^2 = 25$ ✓

(c) $7^{2\log_7 a} = a^2$ ✓

(vii) Volume $= \frac{1}{2} \times \frac{4}{3} \pi r^3$ ✓

$= \frac{2}{3} \pi \times 125$ ✓

$= \frac{250\pi}{3}$ units^3 ✓

14

Question 7

(a) $3^{2-x} = 9^x$

$3^{2-x} = 3^{2x}$ ✓

$2 - x = 2x$

$3x = 2$

$x = \frac{2}{3}$ ✓

(b) $x - y = 10$ — (1)

$\frac{12}{y} - \frac{12}{x} = 5$ — (2)

Rewrite (1) $y = x - 10$ ✓

$\frac{12}{x-10} - \frac{12}{x} = 5$

$x(x-10)$ $12x - 12(x-10) = 5x^2 - 50x$ ✓

$120 = 5x^2 - 50x$

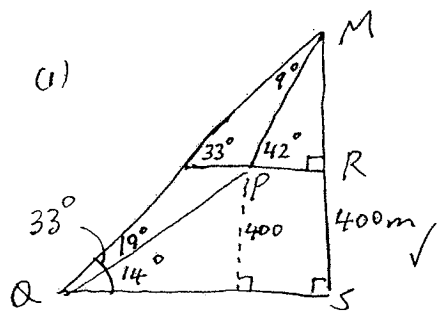
$x^2 - 10x - 24 = 0$ ✓

$(x-12)(x+2) = 0$

$x = 12$ $x = -2$ ✓ x values

$y = 2$ $y = -12$ ✓ y values

(c) (i)



$\angle QMP = 9^\circ$ ✓

(ii) $\frac{400}{QP} = \sin 14^\circ$ ✓

$QP = 400 \div \sin 14^\circ$

$= 1653.43$ ✓

$\frac{MP}{\sin 19^\circ} = \frac{1653.43}{\sin 9^\circ}$

$MP = 3441.07$ ✓

$MR = 3441.07 \sin 42^\circ$

$= 2302.53$ ✓

Height of mountain above sea level

$= 2300 + 400 + 200$ ✓

$= 2900$ metres (nearest 10 metres)

Question 8

(a) $\left(\sqrt[3]{4} \times \frac{1}{\sqrt[6]{8}} \times \sqrt[12]{2^{-1}}\right)^4$

$= \left(2^{\frac{2}{3}} \times 2^{-\frac{1}{2}} \times 2^{-\frac{1}{12}}\right)^4$ ✓

$= 2^{\frac{8}{3}} \times 2^{-2} \times 2^{-\frac{1}{3}} = \sqrt[3]{2}$ ✓

(b) $\log_{10}\left(2^x + \frac{475}{2^x}\right) = 2$

$2^x + \frac{475}{2^x} = 100$ ✓

Let $u = 2^x$, $u^2 - 100u + 475 = 0$ ✓

$(u-95)(u-5) = 0$

$u = 95$ or $u = 5$

So $2^x = 95$ or $2^x = 5$ ✓

Sum $= \log_2 95 + \log_2 5$

$= \log_2 475 = \frac{\log_{10} 475}{\log_{10} 2}$ ✓

(c) (i) $5^4 \times 2^{28} + 2^{32}$

$= 2^{28}(5^4 + 2^4)$ ✓

$= 2^{28} \times 641$

$5^4 \times 2^{28} - 1$

$= (5 \times 2^7)^4 - 1$ ✓

$= (5\times 2^7 - 1)(5\times 2^7 + 1)((5\times 2^7)^2 + 1)$

$= 641 \times (5\times 2^7 + 1)((5\times 2^7)^2 + 1)$

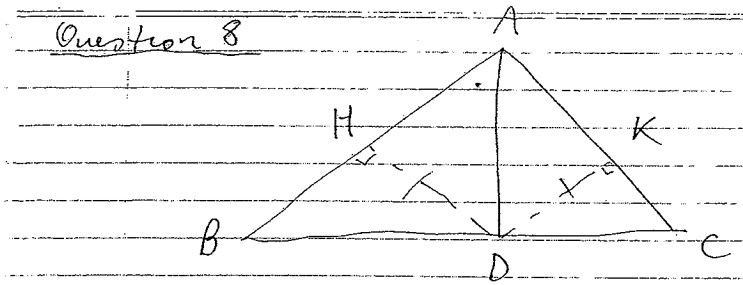
(ii) $5^4 \times 2^{28} + 2^{32} - (5^4 \times 2^{28} - 1)$ ✓

$= 2^{32} + 1$ which is the difference between two numbers divisible by 641

So $2^{32} + 1$ is divisible by 641

So F_5 is not prime

Question 8



(i) The heights are the same

So $\frac{\text{area } \Delta ABD}{\text{area } \Delta ACD} = \frac{BD}{DC}$ ✓

(ii) $\frac{\frac{1}{2} AB \times HD}{\frac{1}{2} AC \times DK} = \frac{BD}{DC}$

But $\frac{AB}{AC} = \frac{BD}{DC}$

$\therefore \frac{HD}{DK} = 1$ ✓

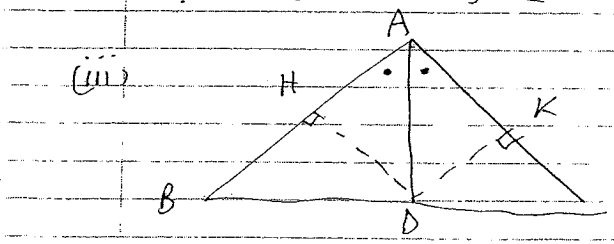
i.e. $HD = DK$

So $\Delta AHD \equiv \Delta AKD$ (RHS test)

$\therefore \angle HAB = \angle KAD$ (matching angles of congruent Δ's) ✓

$\therefore$ AD bisects $\angle BAC$.

(iii)



$HD = DK$ by the (AAS test) ✓

But $\frac{\frac{1}{2} AB \times HD}{\frac{1}{2} AC \times KD} = \frac{BD}{DC}$

$\therefore \frac{AB}{AC} = \frac{BD}{DC}$ So $\frac{AB}{BD} = \frac{AC}{DC}$ ✓