SYDNEY GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT
HALF-YEARLY EXAMINATIONS 2007

FORM V

MATHEMATICS EXTENSION 1

Examination date

Wednesday 9th May 2007

Time allowed

2 hours

Instructions

All nine questions may be attempted.
All nine questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.

Collection

Write your name, class and master clearly on each booklet.
Hand in the nine questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

5A: WMP	5B: PKH	5C: REP
5D: BDD	5E: FMW	5F: GJ
5G: JNC	5H: DS	5I: KWM

Checklist

Folded A3 booklets: 9 per boy. A total of 1500 booklets should be sufficient.
Candidature: 134 boys.

Examiner

JNC

<u>QUESTION ONE</u> (14 marks) Use a separate writing booklet.

(a) Simplify $\sqrt{98}$.

(b) Expand and simplify $(5-\sqrt{2})^2$.

(c) Factorize a^3-1.

(d) Express $\dfrac{2}{3-\sqrt{3}}$ with a rational denominator.

(e) Given that $P(x)=x^2-2x$, find the value of $P(-2)$.

(f) Write down the exact value of $\cos(-45^\circ)$.

(g) Calculate the perpendicular distance from the point $(-1,-1)$ to the line with equation $5x-12y+7=0$.

(h) Evaluate $|-3|-|4-9|$.

(i) Sketch the graph of $y=(x-1)^2$, showing all intercepts with the coordinate axes.

<u>QUESTION TWO</u> (14 marks) Use a separate writing booklet.

(a) Solve $2x^2+3x-9=0$.

(b) Solve $5-x\geq 7$ and graph the solution on a number line.

(c) Express $0{\cdot}\dot{2}\dot{7}$ as a fraction in lowest terms. You MUST show full working.

(d) Write down the value of $\operatorname{cosec} 38^\circ 15'$, correct to four decimal places.

(e) Given the arithmetic sequence $13, 17, 21, \ldots$, find:

(i) the 999th term,

(ii) the sum of the first 999 terms.

(f) Sketch the graph of $(x-2)^2+y^2=4$.

Exam continues next page ...

QUESTION THREE (14 marks) Use a separate writing booklet.

(a) Find $P(1-\sqrt{5})$ if $P(x) = x^2 - 6$.

(b) Find rational numbers x and y such that $x + y\sqrt{3} = \sqrt{363} - 14$.

(c) Consider the function $f(x) = 1 - 3x$.

(i) Find $f^{-1}(x)$.

(ii) Solve $f(x) = f^{-1}(x)$.

(d) (i) Find the vertex of the parabola $y = x^2 - 4x - 8$.

(ii) Hence sketch the graph of $y = x^2 - 4x - 8$. Show the y-intercept.

(e) Describe the transformation by which the graph of a function $y = f(x)$ is transformed to the graph of $y = f(-x)$.

Exam continues overleaf ...

QUESTION FOUR (14 marks) Use a separate writing booklet.

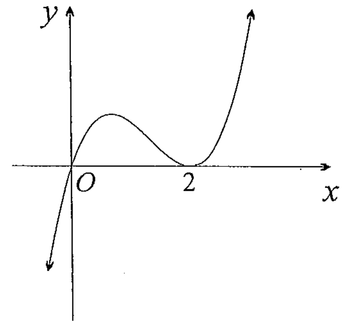
(a) Solve for x:

(i) $(x+2)(x+4) \geq 0$

(ii) $|3x+2| = 12$

(iii) $\dfrac{3}{2-x} < -2$

(b)



In the diagram above, the graph of $y = x(x-2)^2$ is drawn. Use the graph to write down the solution to $x(x-2)^2 > 0$.

(c) A function is defined as $f(x) = \dfrac{2}{1-x^2}$.

(i) Find $f(-x)$.

(ii) Explain why $f(x)$ is an even function.

(d) (i) Sketch the graph of $y = -\sqrt{a^2 - x^2}$, where a is a positive constant.

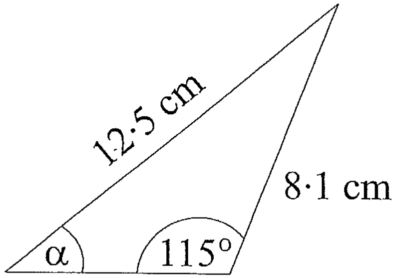
(ii) State the range of $y = -\sqrt{a^2 - x^2}$.

Exam continues next page ...

QUESTION FIVE (14 marks) Use a separate writing booklet.

(a) Evaluate $1 + \tan^2 45°$.

(b)



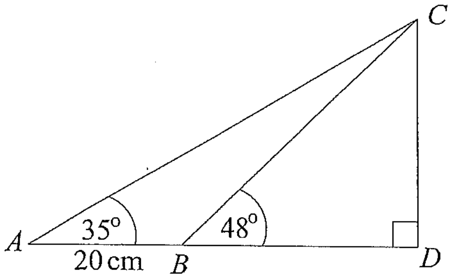
In the diagram above, find α. Give your answer correct to the nearest degree.

(c) Given that $\sin\theta = \frac{8}{15}$, find the possible exact values of $\cos\theta$.

(d) Prove the identity $\dfrac{\cos^2\alpha}{1+\sin\alpha} + \dfrac{\cos^2\alpha}{1-\sin\alpha} = 2$.

(e) Solve $\sqrt{3}\tan\beta + 1 = 0$, for $0° \le \beta \le 360°$.

(f)



In the diagram above, $\angle CDB = 90°$ and $\angle CAB = 35°$. The point B lies on AD such that $AB = 20$ centimetres and $\angle CBD = 48°$.

(i) Show that $BC = \dfrac{20\sin 35°}{\sin 13°}$.

(ii) Hence find BD, correct to the nearest metre.

QUESTION SIX (14 marks) Use a separate writing booklet.

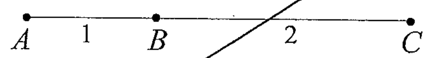
(a) Find the equation of the line which passes through the point $(3, -1)$ and is perpendicular to the line $3x - y = 0$.

(b) Find the acute angle that the line $x\sqrt{3} + y - 7 = 0$ makes with the x-axis.

(c) (i) Write down the general form of a line which passes through T, the point of intersection of the lines $x + 2y - 15 = 0$ and $3x - 2y + 1 = 0$.

(ii) Hence find the equation of the line through T and $(2, 1)$. Give your answer in general form.

(d) (i)



In the diagram above, $2AB = BC$. Write down the ratio in which C divides the interval AB.

(ii) Find the coordinates of C if $A = (-3, -1)$ and $B = (2, -4)$.

(e) The point $(0, 0)$ lies on the line $3x + 4y = 0$. Find the perpendicular distance between the parallel lines $3x + 4y = 0$ and $3x + 4y - 5 = 0$.

QUESTION SEVEN (14 marks) Use a separate writing booklet.

(a) Consider the geometric series $2 - 4 + 8 - \cdots$.

(i) Find the twentieth term.

(ii) Find the sum of the first twenty terms.

(b) How many terms of the series $5 + 8 + 11 + \cdots$ must be taken to give a sum of 608?

(c) (i) State the condition for which the limiting sum of a geometric series exists.

(ii) Find the limiting sum of the series $1{\cdot}47 + 0{\cdot}21 + 0{\cdot}03 + \cdots$.

(d) The terms $2m - 8, 2m + 4$ and $5m - 2$ form a geometric sequence.

(i) Find both possible values of m.

(ii) Write down the sequence for each value of m.

QUESTION EIGHT (14 marks) Use a separate writing booklet.

(a) Consider the function $y = \dfrac{x(x+2)}{(x-2)^2}$.

(i) Find all intercepts with the coordinate axes.

(ii) Write down the equation of the vertical asymptote.

(iii) Find the equation of the horizontal asymptote.

(iv) Find the value of x when $y = 1$.

(v) On about one-third of a page, sketch the graph of the function, showing all the features found in parts (i)-(iv).

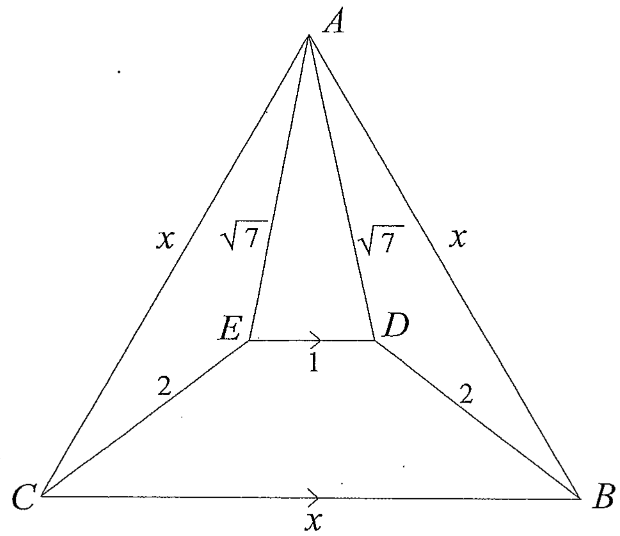
(b) Use mathematical induction to prove that for all positive integers n,

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$

Question Nine follows on the next page.

Exam continues overleaf ...

QUESTION NINE (14 marks) Use a separate writing booklet.



In the diagram above, $\triangle ABC$ is equilateral of side length x cm and $\triangle ADE$ is isosceles such that $AD = AE = \sqrt{7}$. Also $BD = CE = 2$ cm, $DE = 1$ cm and $ED \parallel CB$.
Heron's formula for the area of a triangle is

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where a, b and c are the lengths of the sides of the triangle and $s = \dfrac{a+b+c}{2}$.

(a) Using Heron's formula, or otherwise, show that the area of $\triangle ADE$ is $\dfrac{3\sqrt{3}}{4}$ cm^2.

(b) Hence show that the distance between the parallel lines ED and CB is $\dfrac{(x-3)\sqrt{3}}{2}$ cm..

(c) Find, in terms of x, the area of the trapezium $BCED$.

(d) Show that $\triangle ABD$ has area $\dfrac{x\sqrt{3}}{4}$ cm^2.

(e) Use Heron's formula to show that the area of $\triangle ABD$ is $\frac{1}{4}\sqrt{-x^4 + 22x^2 - 9}$ cm^2.

(f) Hence find the exact value of x in the form $m + n\sqrt{13}$, where m and n are rational numbers.

END OF EXAMINATION

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QUESTION ONE

a) $\sqrt{98} = 7\sqrt{2}$ ✓

b) $(5-\sqrt{2})^2 = 27 - 10\sqrt{2}$ ✓✓

c) $a^3 - 1 = (a-1)(a^2+a+1)$ ✓

d) $\left.\begin{aligned}\frac{2}{3-\sqrt{3}} &= \frac{2}{3-\sqrt{3}} \times \frac{3+\sqrt{3}}{3+\sqrt{3}} \\ &= \frac{2(3+\sqrt{3})}{6}\end{aligned}\right\}$ ✓

$= \frac{1}{3}(3+\sqrt{3})$ ✓

e) $P(-2) = (-2)^2 - 2(-2)$

$= 4 + 4$

$= 8$ ✓

f) $\cos(-45^\circ) = \frac{1}{\sqrt{2}}$ ✓

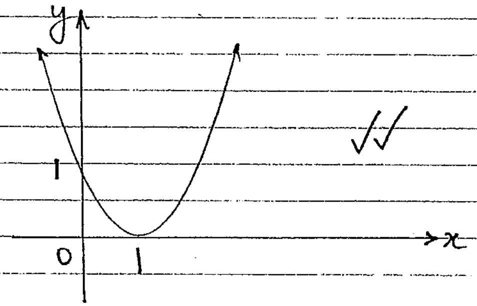
g) $d = \frac{|-5+12+7|}{\sqrt{5^2+12^2}}$ ✓

$= \frac{14}{13}$ ✓

h) $|-3| - |4-9| = 3 - 5$ ✓

$= -2$ ✓

i)



✓✓ Must show intercepts

−1 per error

QUESTION TWO

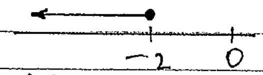
a) $2x^2 + 3x - 9 = 0$

$(2x-3)(x+3) = 0$ ✓

$\therefore x = \frac{3}{2}$ or -3 ✓

b) $5 - x \geq 7$

$x \leq -2$ ✓



✓

c) $0.\dot{2}\dot{7} = 0.272727\ldots$ ✓

Either Let $x = 0.2727\ldots$ ✓

$100x = 27.2727$ ✓

$\left.\begin{aligned}99x &= 27 \\ \therefore x &= \frac{27}{99}\end{aligned}\right\}$ ✓

$= \frac{9}{11}$ ✓

or $x = 0.27 + 0.0027 + 0.000027$

$= \frac{0.27}{1-0.01}$ ✓

$= \frac{0.27}{0.99}$

$= \frac{9}{11}$ ✓

d) $\operatorname{cosec} 38^\circ 15' = 1.6153$ ✓

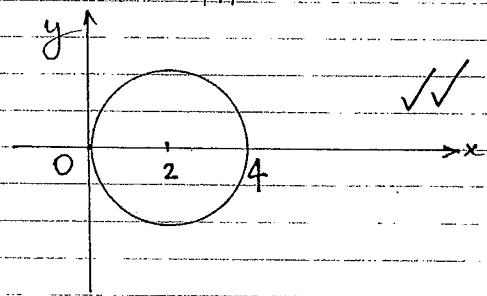
e) (i) $T_{999} = 13 + 998 \times 4$ ✓

$= 4005$ ✓

(ii) $S_{999} = \frac{999}{2}\left[2\times 13 + 998 \times 4\right]$ ✓

$= 2006991$ ✓

f)



✓✓ 1 intercepts

1 shape

QUESTION THREE

a) $P(1-\sqrt{5}) = (1-\sqrt{5})^2 - 6$ ✓

$= 1 - 2\sqrt{5} + 5 - 6$ ✓

$= -2\sqrt{5}$

b) $x = -14$ ✓ and $y\sqrt{3} = \sqrt{363}$ ✓

$= 11\sqrt{3}$ ✓

$\therefore y = 11$ ✓

1 for x
2 for y

c) (i) Let $y = 1-3x$

The inverse is $x = 1-3y$ ✓

$3y = 1-x$

$y = \frac{1-x}{3}$ ✓

So $f^{-1}(x) = \frac{1-x}{3}$

(ii) $1-3x = \frac{1-x}{3}$ ✓

$3-9x = 1-x$

$8x = 2$

$\therefore x = \frac{1}{4}$ ✓

d) (i) $x = \frac{4}{2}$

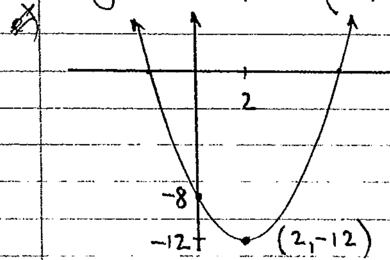
$= 2$ ✓

and $y = 4-8-8$

$= -12$ ✓

So vertex is $(2,-12)$

(ii) y intercept is $(0,-8)$



✓✓

must show y intercept
scale
shape

e) Reflection in the y axis ✓

QUESTION FOUR

a) (i) $(x+2)(x+4) \geq 0$

$x \leq -4$ or $x \geq -2$ ✓✓

(ii) $|3x+2| = 12$

Either $3x+2 = 12$ or $-(3x+2) = 12$ ✓

$\therefore x = \frac{10}{3}$ ✓ $\quad 3x = -14$

$\therefore x = -\frac{14}{3}$ ✓

(iii) $\frac{3}{2-x} < -2$

$3(2-x) < -2(2-x)^2$ ✓

$(2-x)(2(2-x)+3) < 0$ ✓

$(2-x)(7-2x) < 0$ ✓

$2 < x < \frac{7}{2}$ ✓

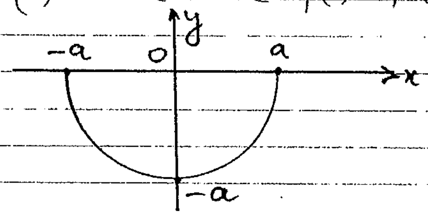
b) $x > 0$ and $x \neq 2$ ✓

c) (i) $f(-x) = \frac{2}{1-(-x)^2}$

$= \frac{2}{1-x^2}$ ✓

(ii) $f(x)$ is even since $f(x) = f(-x)$ ✓

d) (i)



✓✓

(ii) $-a \leq y \leq 0$ ✓

QUESTION FIVE

a) $1+\tan^2 45 = 1+1$

$= 2$ ✓

b) $\dfrac{\sin\alpha}{8.1} = \dfrac{\sin 115}{12.5}$

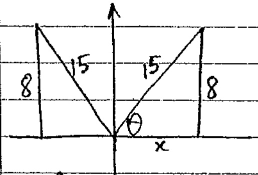
$\sin\alpha = \dfrac{8.1\sin 115}{12.5}$ ✓

$= 0.587287\ldots$

$\therefore\ \alpha = 35.96475\ldots$

$= 36^\circ$ ✓ accept either

c)



$x^2+64 = 225$

$\therefore\ x = \sqrt{161}$ ✓

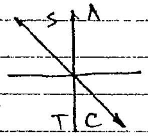
So $\cos\theta = \pm\dfrac{\sqrt{161}}{15}$ ✓

d) LHS $= \dfrac{\cos^2\alpha(1-\sin\alpha)+\cos^2\alpha(1+\sin\alpha)}{(1+\sin\alpha)(1-\sin\alpha)}$ ✓

$= \dfrac{2\cos^2\alpha}{\cos^2\alpha}$

$= 2$ as required. ✓

e) $\tan\beta = -\dfrac{1}{\sqrt{3}}$



related angle $= 30^\circ$ ✓

$\therefore\ \theta = 150^\circ$ or 330° ✓✓

f)(i) $\angle ACB = 48-35$ (exterior angle of A)

$= 13$

$\dfrac{BC}{\sin 35} = \dfrac{20}{\sin 13}$ ✓

$\therefore\ BC = \dfrac{20\sin 35}{\sin 13}$ ✓

(ii) $\dfrac{BD}{BC} = \cos 48$ ✓

$\therefore\ BD = BC\cos 48$

$= 9.4249\ldots$

$= 9$ cm ✓

QUESTION SIX ✓

a) Gradient of line $= 3$, so gradient of required line is $-\frac{1}{3}$.

Equation of line is $y+1 = -\frac{1}{3}(x-3)$ ✓

$3y+3 = -x+3$

$x+3y = 0$. ✓ (any form is acceptable)

b) $\tan\theta = -\sqrt{3}$ ✓

$\therefore\ \theta = 120^\circ$ and acute angle is 60°. ✓

c)(i) $(x+2y-15)+k(3x-2y+1) = 0$ ✓

(ii) $(2+2-15)+k(6-2+1) = 0$

$\therefore\ k = \dfrac{11}{5}$ ✓

$(x+2y-15)+\dfrac{11}{5}(3x-2y+1) = 0$ ✓

$5x+10y-75+33x-22y+11 = 0$

$38x-12y-64 = 0$

$19x-6y-32 = 0$ ✓

d)(i) $3:-2$ or $-3:2$ or $3:2$ externally ✓

(ii) $C = \left(\dfrac{3\times 2+-2\times -3}{3-2}, \dfrac{3\times -4+-2\times -1}{3-2}\right)$

$= (12, -10)$ ✓✓

e) $d = \dfrac{|3\times(0)+4\times(0)-5|}{\sqrt{3^2+4^2}}$ ✓

$= \dfrac{5}{5}$

$= 1$ ✓

QUESTION SEVEN

a) (i) $T_{20} = 2 \times (-2)^{19}$

$= -1048576$ ✓

(ii) $S_{20} = \dfrac{2\left(1-(-2)^{20}\right)}{1-(-2)}$ ✓

$= \dfrac{2}{3}(-1048575)$

$= -699050$ ✓

b) $\dfrac{n}{2}\left[10 + (n-1) \times 3\right] = 608$ ✓

$7n + 3n^2 = 1216$

$3n^2 + 7n - 1216 = 0$ ✓

$(3n+64)(n-19) = 0$

$\therefore n = 19$ or $-\frac{64}{3}$

So $n = 19$ only ✓

c) (i) $|r| < 1$ ✓

(ii) $S_\infty = \dfrac{1.47}{1 - \frac{1}{7}}$ ✓

$= \dfrac{7 \times 1.47}{6}$ ✓

$= \dfrac{1029}{600}$ or $\dfrac{10.29}{6}$

$= 1.715$ or $\dfrac{343}{200}$ ✓

d) $\dfrac{2m+4}{2m-8} = \dfrac{5m-2}{2m+4}$ ✓

(i) $(2m+4)^2 = (5m-2)(2m-8)$

$4m^2 + 16m + 16 = 10m^2 - 44m + 16$

$6m^2 - 60m = 0$ ✓

$6m(m-10) = 0$

$\therefore m = 0$ or 10 ✓

(ii) So ~~ratios~~ sequences are $-8, 4, -2, \ldots$

and $12, 24, 48, \ldots$ ✓

QUESTION EIGHT

a) (i) When $x = 0$, $y = 0$ and when $y = 0$, $x = -2$.

So intercepts are $(0,0)$ and $(-2,0)$

(ii) Vertical asymptote is $x = 2$

(iii) $f(x) = \dfrac{x^2 + 2x}{x^2 - 4x + 4}$

$= \dfrac{1 + \frac{2}{x}}{1 - \frac{4}{x} + \frac{4}{x^2}}$

So $\lim\limits_{x \to \infty} f(x) = \dfrac{1+0}{1-0+0} = 1$, and horizontal asymptote is $y = 1$

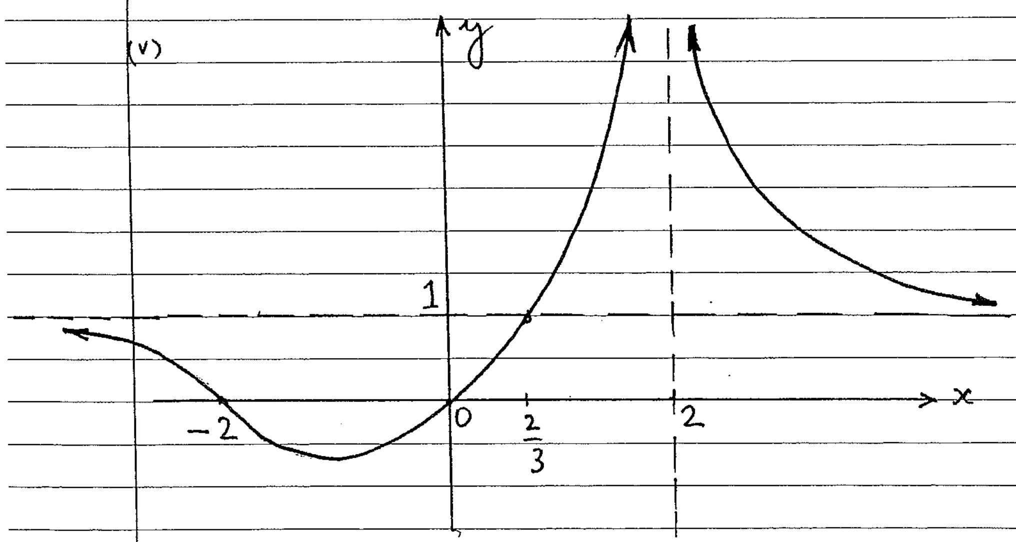
(iv) $\dfrac{x(x+2)}{(x-2)^2} = 1$

$x^2 + 2x = x^2 - 4x + 4$

$6x = 4$

$\therefore x = \dfrac{2}{3}$

(v)



b) When $n=1$, $LHS = \frac{1}{1\times 3} = \frac{1}{3}$ and $RHS = \frac{1}{2\times 1 + 1} = \frac{1}{3}$

So the result holds for $n=1$

Suppose that the result holds for $n=k$, where k is an integer.

ie/ $\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$ —**

We need to prove the result for $n=k+1$.

ie/ $\frac{1}{1\times 3} + \frac{1}{3\times 5} + \cdots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$

$LHS = \frac{k}{(2k+1)} + \frac{1}{(2k+1)(2k+3)}$, by the induction hypothesis

$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$

$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$

$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$

$= \frac{k+1}{2k+3}$

$= RHS$

It follows by mathematical induction that the result is true for all cardinals n.

QUESTION NINE

a) $S = \frac{1+2\sqrt{7}}{2}$ ✓

Area $\Delta ADE = \sqrt{\left(\frac{1+2\sqrt{7}}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{2\sqrt{7}-1}{2}\right)}$

$= \frac{1}{2}\sqrt{\frac{27}{4}}$ ✓

$= \frac{3\sqrt{3}}{4}$ ✓

b) Let h be $\perp$ height of ΔADE and H be $\perp$ distance between parallel lines

So $\frac{1}{2}\times 1 \times h = \frac{3\sqrt{3}}{4}$

$\therefore h = \frac{3\sqrt{3}}{2}$ ✓

The area of ΔABC is: $\frac{1}{2}x\left(H + \frac{3\sqrt{3}}{2}\right) = \frac{1}{2}x^2\sin 60^\circ$ ✓

$H = \frac{\sqrt{3}}{2}x - \frac{3\sqrt{3}}{2}$

$= \frac{(x-3)\sqrt{3}}{2}$ ✓

c) Area of $BCED = \frac{1}{2}\times\frac{(x-3)\sqrt{3}}{2}(x+1)$

$= \frac{\sqrt{3}}{4}(x^2-2x-3)$ ✓

d) $2\times\Delta ABD = \Delta ABC - \Delta ADE - BCED$

$= \frac{1}{2}x^2\sin 60 - \frac{3\sqrt{3}}{4} - \frac{\sqrt{3}}{4}(x^2-2x-3)$ ✓

$= \frac{\sqrt{3}}{4}(x^2-3-x^2+2x+3)$ ✓

$= \frac{x\sqrt{3}}{2}$

$\therefore$ area $\Delta ABD = \frac{x\sqrt{3}}{4}$ ✓

e) $S = \frac{1}{2}(x+2+\sqrt{7})$

$$\Delta ABD = \sqrt{\frac{1}{2}(x+2+\sqrt{7})\frac{1}{2}(x+2-\sqrt{7})\frac{1}{2}(x-2+\sqrt{7})\frac{1}{2}(-x+2+\sqrt{7})}$$

$$= \frac{1}{4}\sqrt{\{(x+2)^2-7\}\{7-(x-2)^2\}}$$

$$= \frac{1}{4}\sqrt{(x^2+4x-3)(3+4x-x^2)}$$

$$= \frac{1}{4}\sqrt{22x^2-x^4-9}$$

f) Equating (d) and (e)

$$x\sqrt{3} = \sqrt{22x^2-x^4-9}$$

$$x^4-19x^2+9 = 0$$

$$x^2 = \frac{19 \pm \sqrt{325}}{2}$$

$$x^2 = \frac{19 \pm 5\sqrt{13}}{2}$$

Since $x > 1$, $x^2 = \frac{19+5\sqrt{13}}{2}$ only

Consider $\frac{19+5\sqrt{13}}{2} = \frac{38+10\sqrt{13}}{4}$

$$= \frac{25+10\sqrt{13}+13}{4}$$

$$= \left(\frac{5+\sqrt{13}}{2}\right)^2$$

$$\therefore x = \frac{5+\sqrt{13}}{2}$$