SYDNEY GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT
HALF-YEARLY EXAMINATIONS 2008

FORM V

MATHEMATICS EXTENSION 1

Examination date

Wednesday 14th May 2008

Time allowed

2 hours

Instructions

All eight questions may be attempted.
All eight questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.

Collection

Write your name, class and master clearly on each booklet.
Hand in the eight questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

5A: DS	5B: MLS	5C: DNW	5D: TCW
5E: PKH	5F: SJE	5G: FMW	5H: BDD

Checklist

Folded A3 booklets: 8 per boy. A total of 1250 booklets should be sufficient.
Candidature: 128 boys.

Examiner

TCW

QUESTION ONE (15 marks) Use a separate writing booklet.

(a) Simplify:

(i) $\sqrt{72}-\sqrt{50}$

(ii) $3\sqrt{10}\times 4\sqrt{15}$

(iii) $(2-\sqrt{3})^2$

(b) (i) Evaluate $|2^2-3^2|$.

(ii) Solve $|x-2|=4$.

(c) (i) Find the exact value of $\sin 225^\circ$.

(ii) Find $\sec 333^\circ 11'$ correct to 2 decimal places.

(d) Factorise a^3+1.

(e) Write down the domain of $y=\sqrt{x}$.

(f) Complete the squares to find the centre and the radius of the circle with equation

$$x^2-4x+y^2+12y=104\,.$$

QUESTION TWO (15 marks) Use a separate writing booklet.

(a) (i) If $10-\sqrt{b}=10-2\sqrt{5}$, find b.

(ii) Rationalise the denominator then simplify $\dfrac{2}{4+3\sqrt{2}}$.

(b) Consider the straight line with equation $x+\sqrt{3}y-3\sqrt{3}=0$.

(i) Find the y-intercept.

(ii) Find the angle of inclination to the positive direction of the x-axis.

(c) Solve $|3x+9|<12$.

(d) (i) Sketch the graph of $y=x^2-9$, clearly showing all the intercepts with the axes.

(ii) Solve $x^2-9<0$.

(e) (i) Sketch the graphs of $y=|3x|$ and $2x-y-1=0$ on a single number plane, clearly showing all the intercepts with the axes.

(ii) For what values of ℓ will the equation $2x+\ell=|3x|$ have at least one solution?

Exam continues next page ...

QUESTION THREE (15 marks) Use a separate writing booklet.

(a) Write $0{\cdot}0\dot{3}\dot{4}$ as a fraction in lowest terms. Clearly show your working.

(b) The point A is $(3, -5)$ and the point B is $(-1, -3)$.

(i) The point P divides the interval AB internally in the ratio $1 : 3$. Find the coordinates of P.

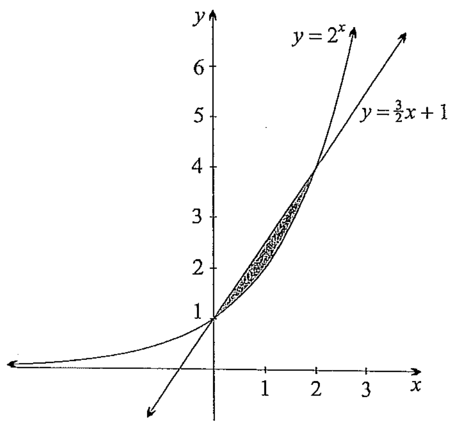
(ii) The point Q divides the interval AB externally in the ratio $3 : 1$. Find the coordinates of Q.

(iii) Find the equation of the perpendicular bisector of AB.

(c) (i) Show that the perpendicular distance from the point $(-2, 4)$ to the line $3x - y - 10 = 0$ is $2\sqrt{10}$ units.

(ii) Hence, or otherwise, determine the number of times that the line $3x - y - 10 = 0$ intersects the circle $(x + 2)^2 + (y - 4)^2 = 40$.

(d)



The diagram above shows the graphs of $y = 2^x$ and $y = \frac{3}{2}x + 1$, which intersect at the points $(0, 1)$ and $(2, 4)$. The region bounded by the two graphs is shaded.

(i) Use the diagram to solve $\frac{3}{2}x + 1 < 2^x$.

(ii) Use inequations to describe the shaded region in the diagram above.

QUESTION FOUR (15 marks) Use a separate writing booklet.

(a) The nth term of a sequence is given by $T_n = 13n - 15$.

(i) Show that the sequence is arithmetic.

(ii) Find the first 3 terms of the sequence.

(iii) Is 2651 a term of the sequence? Show your working.

(b) Consider the arithmetic series $180 + 168 + 156 + \cdots$.

(i) Show that $S_n = 6n(31 - n)$.

(ii) Hence find two values of n such that the sum of the first n terms is 1104.

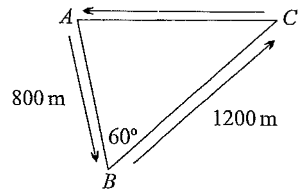
(c) Evaluate $\displaystyle\sum_{n=2}^{5} 10 \times (-2)^n$.

(d) The fifth term of a geometric sequence is 224 and the tenth term is -7168. Find the common ratio.

(e) Find the limiting sum of the geometric series $1 + \frac{3}{5} + \frac{9}{25} + \cdots$.

QUESTION FIVE (15 marks) Use a separate writing booklet.

(a)



Henry completes an ocean swim by following one lap of the triangular course shown above, swimming in the direction indicated. The swim starts and finishes at point A.

(i) Find the exact area enclosed by the triangular course.

(ii) Use the cosine rule to show that when Henry reaches point C he still has approximately 1058 metres to go.

(iii) Use the sine rule to find, correct to the nearest degree, the size of $\angle A$, given that it is acute.

(b) Prove that $\dfrac{\sin\theta}{1 - \sin\theta} - \dfrac{\sin\theta}{1 + \sin\theta} = 2\tan^2\theta$.

(c) Given that $\sin\theta = -\frac{1}{4}$ and $\cos\theta > 0$, draw a quadrant diagram showing the angle θ, then find the exact value of $\cot\theta$.

(d) Solve $\tan 2x = 1$, for $0° \leq x \leq 360°$.

QUESTION SIX (15 marks) Use a separate writing booklet.

(a) Solve $\dfrac{25}{x+5} \geq 3$.

(b) The straight lines with equations $2x - 5y + 10 = 0$ and $3x - 4y - 8 = 0$ intersect at the point L.

(i) Show that the line $2x - 5y + 10 + k(3x - 4y - 8) = 0$ has gradient $\dfrac{2+3k}{5+4k}$.

(ii) Without finding the coordinates of L, find the equation of the line through L with gradient $\frac{1}{2}$. Give your answer in general form.

(c) Consider the series $1{\cdot}01 + 1{\cdot}01^2 + 1{\cdot}01^3 + \cdots$.

(i) Show that if the sum to n terms of the series exceeds one million, then

$$1{\cdot}01^n > \frac{1\,000\,000}{101} + 1.$$

(ii) Hence find the least number of terms of the series that must be added for the sum to exceed one million.

(d) Solve $3\cos\theta + \cot\theta = 0$, for $0^\circ \leq \theta \leq 360^\circ$. Give your solutions correct to the nearest minute where necessary.

Exam continues overleaf ...

QUESTION SEVEN (15 marks) Use a separate writing booklet.

(a) Consider the function $f(x) = \dfrac{4-x}{1+x}$.

(i) Write down the domain of $f(x)$.

(ii) Show that $f(x) = f^{-1}(x)$.

(iii) Show that $f(x) = \dfrac{5}{1+x} - 1$.

(iv) Describe a pair of transformations by which the graph of the hyperbola $y = \dfrac{5}{x}$ is transformed to the graph of $y = f(x)$.

(v) Hence, or otherwise, sketch $y = f(x)$, showing all the intercepts with the axes and both asymptotes.

(vi) Explain how the graph demonstrates that $y = f(x)$ is its own inverse.

(b) Consider the function $y = \dfrac{x^2}{1+x^2}$.

(i) Find any intercepts with the axes.

(ii) Determine whether the function is even, odd or neither. Show your working.

(iii) Find x when $y = \frac{1}{2}$.

(iv) Find the equation of the horizontal asymptote. Show your working.

(v) Sketch the graph of the function showing all the above features.

(vi) Write down the range of the function.

Exam continues next page ...

QUESTION EIGHT (15 marks) Use a separate writing booklet.

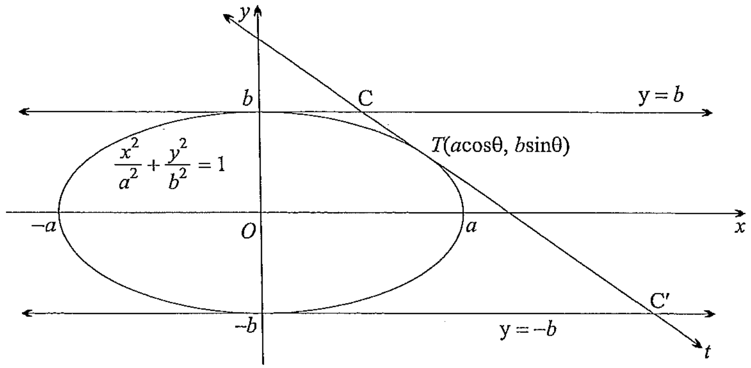
(a) Use mathematical induction to prove that $13 \times 6^n + 2$ is divisible by 5, for all positive integers n.

(b) For what values of x does the geometric series

$$1 + 2\cos x + 4\cos^2 x + 8\cos^3 x + \cdots$$

have a limiting sum, if $0° \leq x \leq 360°$?

(c)



When the unit circle $x^2+y^2 = 1$ is stretched vertically by a factor of b and horizontally by a factor of a, it is transformed to an ellipse with equation $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$.

The diagram above shows the tangent t to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $T(a\cos\theta, b\sin\theta)$. The gradient of the tangent t is $-\frac{b}{a}\cot\theta$. The ellipse has horizontal tangents at its y-intercepts, with equations $y = b$ and $y = -b$ as shown above. The origin is O.

(i) Show that the tangent t has equation $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$.

(ii) The tangent t intersects the horizontal tangents to the ellipse at the points C and C', as shown in the diagram. Show that

$$C = (a\sec\theta - a\tan\theta, b) \qquad \text{and} \qquad C' = (a\sec\theta + a\tan\theta, -b).$$

(iii) The circle with diameter CC' intersects the x-axis at D and D'. Show that

$$OD \times OD' = a^2 - b^2.$$

END OF EXAMINATION

V EXTENSION 1 – HALF-YEARLY EXAMINATION

QUESTION 1

(a) (i) $\sqrt{72} - \sqrt{50} = 6\sqrt{2} - 5\sqrt{2}$ ✓

$= \sqrt{2}$ ✓

(ii) $3\sqrt{10} \times 4\sqrt{15} = 12\sqrt{150}$ ✓

$= 12 \times 5\sqrt{6}$

$= 60\sqrt{6}$ ✓

(iii) $(2-\sqrt{3})^2 = 4 - 4\sqrt{3} + 3$ ✓

$= 7 - 4\sqrt{3}$ ✓

(b) (i) $|2^2 - 3^2| = |4-9|$

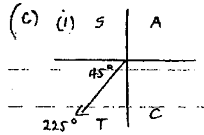
$= 5$ ✓

(ii) $|x-2| = 4$

$x - 2 = 4$ or $x - 2 = -4$

$x = 6$ or $x = -2$ ✓

(c) (i)



$\sin 225° = -\frac{1}{\sqrt{2}}$ ✓

(ii) $\sec 333°11' \doteqdot 1{\cdot}12$ (2dp) ✓

[NO ROUNDING PENALTY]

(d) $a^3 + 1 = (a+1)(a^2 - a + 1)$ ✓

(e) D: $x \geq 0$ ✓

(f) $x^2 - 4x + 4 + y^2 + 12y + 36 = 104 + 40$ ✓

$(x-2)^2 + (y+6)^2 = 144$ ✓

centre $= (2, -6)$ radius $= 12$ units ✓

15

QUESTION 2

(a) (i) $10 - \sqrt{b} = 10 - 2\sqrt{5}$

$10 - \sqrt{b} = 10 - \sqrt{20}$

$b = 20$ ✓

(ii) $\dfrac{2}{4+3\sqrt{2}} = \dfrac{2}{4+3\sqrt{2}} \times \dfrac{4-3\sqrt{2}}{4-3\sqrt{2}}$

$= \dfrac{2(4-3\sqrt{2})}{16-18}$ ✓

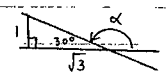
$= 3\sqrt{2} - 4$ ✓

(b) $x + \sqrt{3}y - 3\sqrt{3} = 0$

$y = -\frac{1}{\sqrt{3}}x + 3$

(i) y-intercept is 3 ✓
(or coordinates $(0, 3)$)

(ii)



angle of inclination α:

$\tan\alpha = -\frac{1}{\sqrt{3}}$ ✓

$\alpha = 150°$ ✓

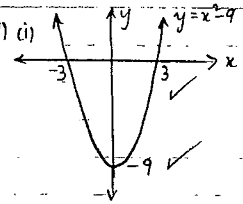
(c) $|3x+9| < 12$

$-12 < 3x + 9 < 12$ ✓

$-21 < 3x < 3$

$-7 < x < 1$ ✓

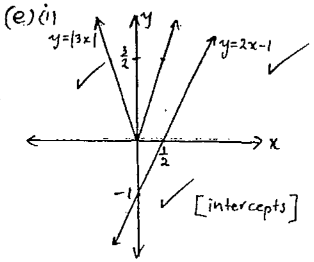
(d) (i)



(ii) $x^2 - 9 < 0$

$-3 < x < 3$ ✓

(e) (i)



(ii) $2x + \ell = |3x|$

For at least one solution:

$\ell \geq 0$ ✓

15

QUESTION 3

(a) Let $x = 0.0343434\ldots$

$100x = 3.4343434\ldots$

$99x = 3.4$

$x = \frac{3.4}{99}$

$x = \frac{34}{990}$

So $0.0\dot{3}\dot{4} = \frac{17}{495}$

R $0.0\dot{3}\dot{4} = 0.034 + 0.00034 + 0.0000034 + \ldots$

$= \frac{0.034}{1-0.01}$

$= \frac{34}{990}$

$= \frac{17}{495}$

(b) (i) $A(3,-5)$ $B(-1,-3)$

$1:3$

$P = \left(\frac{3\times3-1}{4}, \frac{-15-3}{4}\right)$

$= \left(2, -4\frac{1}{2}\right)$

(ii) $A(3,-5)$ $B(-1,-3)$

$-3:1$

$Q = \left(\frac{3+3}{-2}, \frac{-5+9}{-2}\right)$

$= (-3,-2)$

(iii) MIDPOINT $M_{AB} = \left(\frac{3-1}{2}, \frac{-5-3}{2}\right)$

$= (1,-4)$

GRADIENT $m_{AB} = \frac{-3+5}{-1-3}$

$= -\frac{1}{2}$

$\perp$ BISECTOR OF AB:

$y+4 = 2(x-1)$ [GRADIENT]

$y = 2x-6$

(c) (i) $\perp$ distance $= \frac{|3(-2)-4-10|}{\sqrt{3^2+(-1)^2}}$

$= \frac{20}{\sqrt{10}} \times \frac{\sqrt{10}}{\sqrt{10}}$

$= 2\sqrt{10}$ units

(ii) circle: $(x+2)^2 + (y-4)^2 = (2\sqrt{10})^2$

so radius = $\perp$ distance from centre to line

$y = 3x-10$ is a tangent to the circle and intersects it only once.

(d) (i) $x < 0$ or $x > 2$

(ii) $y \leq \frac{3}{2}x+1$ and $y \geq 2^x$

15

QUESTION 4

(a)

(i) $T_n - T_{n-1} = 13n-15-(13(n-1)-15)$

$= 13n-15-13n+28$

$= 13$

So the sequence is arithmetic with $d = 13$.

(ii) $T_n = 13n-15$

$T_1 = 13-15$

$= -2$

$T_2 = 26-15$

$= 11$

$T_3 = 39-15$

$= 24$

(iii) Let $T_n = 2651$

$13n-15 = 2651$

$13n = 2666$

$n = 205\frac{1}{13}$

n must be a positive integer, so 2651 is not a term in the sequence.

(b) (i) $180+168+156+\ldots$

AP: $a=180$, $d=-12$

$S_n = \frac{n}{2}(360+(n-1)(-12))$

$= \frac{n}{2}(372-12n)$

$= 6n(31-n)$

(ii) Let $S_n = 1104$

$6n(31-n) = 1104$

$-6n^2+186n-1104 = 0$

$n^2-31n+184 = 0$

$(n-23)(n-8) = 0$

$n = 8$ or 23

(c) $\sum_{n=2}^{5} 10\times(-2)^n = 10\times4 - 10\times8 + 10\times16 - 10\times32$

$= 40-80+160-320$

$= -200$

(d) GP: $T_5 = 224$

$ar^4 = 224$ —— ①

$T_{10} = -7168$

$ar^9 = -7168$ —— ②

② ÷ ①: $\frac{ar^9}{ar^4} = -\frac{7168}{224}$

$r^5 = -32$

$r = -2$

(e) $1 + \frac{3}{5} + \frac{9}{25} + \ldots$

GP: $a=1$, $r = \frac{3}{5}$

$S_\infty = \frac{a}{1-r}$

$= \frac{1}{1-\frac{3}{5}}$

$= \frac{1}{\frac{2}{5}}$

$= 2\frac{1}{2}$

15

QUESTION 5

(i) $A = \frac{1}{2} \times 800 \times 1200 \times \sin 60°$ ✓

$= 480\,000 \times \frac{\sqrt{3}}{2}$

$= 240\,000\sqrt{3}\ m^2$ ✓

(ii) $AC^2 = 800^2 + 1200^2 - 2 \times 800 \times 1200 \times \cos 60°$ ✓

$AC^2 = 640\,000 + 1440\,000 - 960\,000$

$AC^2 = 1120\,000$ ✓

$AC = 400\sqrt{7}\ m$

$AC \doteqdot 1058\ m$ (nearest metre)

(iii) $\frac{\sin A}{1200} = \frac{\sin 60°}{400\sqrt{7}}$ ✓

$\sin A = \frac{1200 \times \frac{\sqrt{3}}{2}}{400\sqrt{7}}$

$\sin A = \frac{3\sqrt{3}}{2\sqrt{7}}$ ✓

$\angle A \doteqdot 79°$ (nearest degree) ✓

) LHS $= \frac{\sin\theta}{1-\sin\theta} - \frac{\sin\theta}{1+\sin\theta}$

$= \frac{\sin\theta(1+\sin\theta) - \sin\theta(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$ ✓

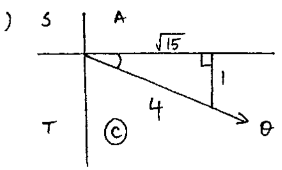
$= \frac{\sin\theta + \sin^2\theta - \sin\theta + \sin^2\theta}{1-\sin^2\theta}$

$= \frac{2\sin^2\theta}{\cos^2\theta}$ ✓

$= 2\tan^2\theta$

$=$ RHS

)



✓

$\cot\theta = -\sqrt{15}$ ✓✓

(d) $\tan 2x = 1,\quad 0° \leqslant x \leqslant 360°$

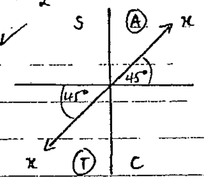
$\tan 2x = 1,\quad 0° \leqslant 2x \leqslant 720°$

$2x = 45°, 225°, 405°$ or $585°$ ✓

$x = 22\frac{1}{2}°, 112\frac{1}{2}°, 202\frac{1}{2}°$ or $292\frac{1}{2}°$

✓ related ∠ = 45°

✓



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QUESTION 6

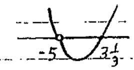
(a) $\frac{25}{x+5} \geqslant 3$

$25(x+5) \geqslant 3(x+5)^2,\quad x \neq -5$ ✓

$3(x+5)^2 - 25(x+5) \leqslant 0,\quad x \neq -5$

$(x+5)(3(x+5) - 25) \leqslant 0,\quad x \neq -5$

$(x+5)(3x-10) \leqslant 0,\quad x \neq -5$ ✓



$-5 < x \leqslant 3\frac{1}{3}$ ✓

b) (i) $2x - 5y + 10 + k(3x - 4y - 8) = 0$

$2x - 5y + 10 + 3kx - 4ky - 8k = 0$

$(2+3k)x - (5+4k)y + 10 - 8k = 0$ ✓

$y = \frac{2+3k}{5+4k}x + \frac{10-8k}{5+4k}$ ✓

so gradient is $\frac{2+3k}{5+4k}$

(ii) when $\frac{2+3k}{5+4k} = \frac{1}{2}$

$4 + 6k = 5 + 4k$

$2k = 1$

$k = \frac{1}{2}$ ✓

Equation of line with gradient $\frac{1}{2}$:

$2x - 5y + 10 + \frac{1}{2}(3x - 4y - 8) = 0$ ✓

$4x - 10y + 20 + 3x - 4y - 8 = 0$

$7x - 14y + 12 = 0$ ✓

[GENERAL FORM]

(c) $1.01 + 1.01^2 + 1.01^3 + \ldots$

GP: $a = 1.01,\ r = 1.01$

(i) $S_n = \frac{1.01(1.01^n - 1)}{1.01 - 1}$ ✓

$= 101(1.01^n - 1)$

If $S_n > 1\,000\,000$

$101(1.01^n - 1) > 1\,000\,000$

$1.01^n - 1 > \frac{1\,000\,000}{101}$ ✓

$1.01^n > \frac{1\,000\,000}{101} + 1$

as required

(ii) solving (i):

$n > \frac{\log_{10}\left(\frac{1\,000\,000}{101} + 1\right)}{\log_{10} 1.01}$ ✓

$n > 924.64$ (2dp)

so at least 925 terms must be added to obtain a sum greater than one million ✓

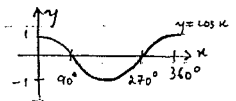
[OR USE TRIAL AND ERROR TO FIND $n = 925$]

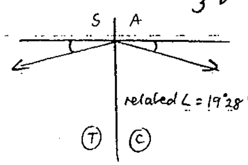
(d) $3\cos\theta + \cot\theta = 0,\quad 0° \leqslant \theta \leqslant 360°$

$3\cos\theta + \frac{\cos\theta}{\sin\theta} = 0$

$\cos\theta\left(3 + \frac{1}{\sin\theta}\right) = 0$ ✓

$\cos\theta = 0$ or $\sin\theta = -\frac{1}{3}$ ✓





related ∠ = 19°28'

$\theta = 90°, 199°28', 270°$ or $340°32'$ ✓

15

QUESTION 7

(a) (i) $D: x \neq -1$ ✓

(ii) $f: \quad y = \dfrac{4-x}{1+x}$

$f^{-1}: \quad x = \dfrac{4-y}{1+y}$

$x + xy = 4 - y$

$y + xy = 4 - x$

$y(1+x) = 4 - x$

$y = \dfrac{4-x}{1+x}$ ✓

(iii) $f(x) = \dfrac{4-x}{1+x}$

$= \dfrac{4-(1+x)+1}{1+x}$

$= \dfrac{5}{1+x} - \dfrac{1+x}{1+x}$ ✓

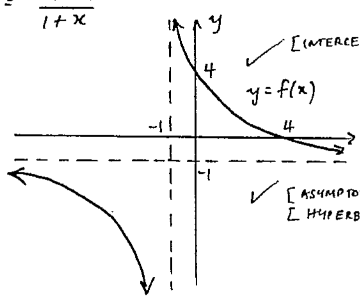
$= \dfrac{5}{1+x} - 1$

(iv) $y = f(x)$ is the hyperbola $y = \frac{5}{x}$ shifted left 1 unit and down 1 unit. ✓

(v) $y = \dfrac{5}{1+x} - 1$ or $y = \dfrac{4-x}{1+x}$

when $x = 0$, $y = 4$

when $y = 0$, $x = 4$



(vi) $y = f(x)$ has reflectional symmetry in the line $y = x$. ($y = x$ is an axis of symmetry). ✓

(b) (i) $y = \dfrac{x^2}{1+x^2}$

when $x = 0$, $y = \dfrac{0}{1+0} = 0$ ✓

when $y = 0$, $\dfrac{x^2}{1+x^2} = 0$

$x = 0$

(ii) Let $f(x) = \dfrac{x^2}{1+x^2}$

$f(-x) = \dfrac{(-x)^2}{1+(-x)^2}$

$= \dfrac{x^2}{1+x^2}$

$= f(x)$ ✓

so $f(x)$ is an even function.

(iii) when $y = \frac{1}{2}$, $\dfrac{x^2}{1+x^2} = \dfrac{1}{2}$

$x^2 = 1$

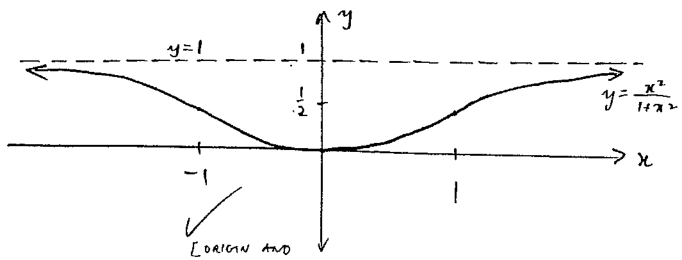
$x = 1$ or -1 ✓

(iv) $y = \dfrac{1}{\frac{1}{x^2} + 1}$ ✓

as $x \to \pm\infty$, $y \to 1$ since $\frac{1}{x^2} \to 0$.

so $y = 1$ is the horizontal asymptote. ✓

(v)



(points of inflexion at $x = \pm\frac{1}{\sqrt{3}}$, $y = \frac{1}{4}$)

(vi) Range: $0 \leqslant y < 1$ ✓

15

QUESTION 8

(a) A. when $n=1$, $13\times 6^n + 2 = 13\times 6 + 2$
$= 80$
$= 5\times 16$ ✓

so the statement is true for $n=1$.

B. suppose that the statement is true for a positive integer k

ie suppose $13\times 6^k + 2 = 5M$, for some integer M ✓

or $13\times 6^k = 5M - 2$ ——— ①

we prove that the statement is true for $n=k+1$.

ie we prove $13\times 6^{k+1} + 2$ is divisible by 5.

$$\begin{aligned} 13\times 6^{k+1} + 2 &= (13\times 6^k)\times 6^1 + 2 \\ &= (5M-2)\times 6 + 2 \quad \text{✓ from ①} \\ &= 6(5M) - 12 + 2 \\ &= 5(6M) - 10 \\ &= 5(6M-2) \quad \text{✓ which is divisible} \end{aligned}$$

by 5 since $6M-2$ is an integer.

C. It follows from parts A and B by mathematical induction that the statement is true for all positive integers n.

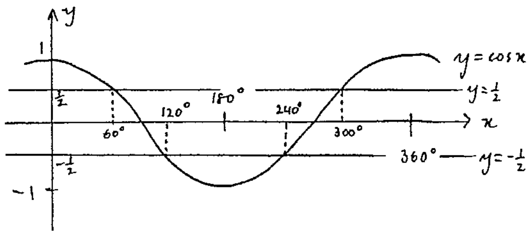
(NO MARK FOR CONCLUSION)

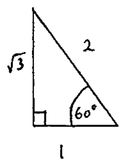
(b) $1 + 2\cos x + 4\cos^2 x + 8\cos^3 x + \ldots$

GP: $r = 2\cos x$, $a = 1$ ✓ (common ratio)

For a limiting sum $-1 < 2\cos x < 1$

$-\frac{1}{2} < \cos x < \frac{1}{2}$ ✓ and $0° \leq x \leq 360°$





solution: $60° < x < 120°$ or $240° < x < 300°$ ✓ ✓

(c) (i) $y - y_1 = m(x - x_1)$

tangent t: $y - b\sin\theta = -\frac{b}{a}\cot\theta\,(x - a\cos\theta)$ ✓

$$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$$

$$bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta)$$

$$\frac{bx\cos\theta}{ab} + \frac{ay\sin\theta}{ab} = \frac{ab}{ab}$$ ✓

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

(ii) tangent t: when $y = b$, $\frac{x}{a}\cos\theta + \frac{b}{b}\sin\theta = 1$

$$\frac{x}{a}\cos\theta = 1 - \sin\theta$$

$$x = \frac{a - a\sin\theta}{\cos\theta}$$

$$x = \frac{a}{\cos\theta} - \frac{a\sin\theta}{\cos\theta}$$

$$x = a\sec\theta - a\tan\theta$$

so $C = (a\sec\theta - a\tan\theta, b)$ ✓

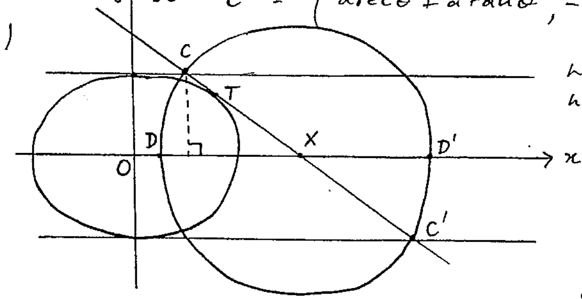
when $y = -b$, $\frac{x}{a}\cos\theta = 1 + \sin\theta$

$$x = \frac{a + a\sin\theta}{\cos\theta}$$

$$x = a\sec\theta + a\tan\theta$$

so $C' = (a\sec\theta + a\tan\theta, -b)$ ✓

(iii)



Let circle, diameter CC', have centre X.

$X = (a\sec\theta, 0)$ ✓

$CX = XD = XD'$

$$CX^2 = b^2 + (a\sec\theta - (a\sec\theta - a\tan\theta))^2$$

$$CX^2 = b^2 + a^2\tan^2\theta$$

$$CX = \sqrt{b^2 + a^2\tan^2\theta}$$ ✓

Now
$$\begin{aligned} OD\times OD' &= (a\sec\theta - CX)(a\sec\theta + CX) \\ &= a^2\sec^2\theta - CX^2 \\ &= a^2\sec^2\theta - b^2 - a^2\tan^2\theta \\ &= a^2(\sec^2\theta - \tan^2\theta) - b^2 \quad \text{✓} \\ &= a^2 - b^2 \end{aligned}$$

$\boxed{1 + \tan^2\theta = \sec^2\theta \quad \sec^2\theta - \tan^2\theta = 1}$