SYDNEY GRAMMAR SCHOOL **YEARLY EXAMINATION 2001**

3 UNIT MATHEMATICS FORM V

Time allowed: 3 hours **Exam date:** 18th October 2001

Instructions:

All questions may be attempted.
All questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.

Collection:

Each question will be collected separately.
Start each question on a new Answer Booklet.
If you use a second booklet for a question, place it inside the first. Do not staple.
Write your name, class and master's initials on each answer booklet:

5A: WMP	5B: JNC	5C: DNW	5D: REN
5E: KWM	5F: BDD	5G: FMW	5H: TCW

QUESTION ONE (Start a new Answer Booklet)

(a) Solve $\tan\theta = 1$, for $0^\circ \le \theta \le 360^\circ$.

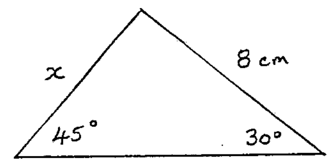
(b) Simplify $\dfrac{x^3-8}{x^2-4}$.

(c) (i) Show that there are 21 terms in the arithmetic series $-2+1+4+\cdots+58$.

(ii) Hence or otherwise find the sum.

(d) Solve the inequation $|x+1| < 2$.

(e)



Find the exact value of x in the diagram above.

(f) (i) Find the gradient of the chord joining the points $A(1,0)$ and $B(2,4)$ on the curve $y = 2x^2 - 2x$.

(ii) Find the gradient of the tangent to the curve $y = 2x^2 - 2x$ at the point $B(2,4)$.

(g) Consider the function $f(x) = x^2 + 3x$.

(i) Show that $f(x+h) - f(x) = 2xh + h^2 + 3h$.

(ii) Use the definition $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ to find $f'(x)$.

Exam continues next page ...

QUESTION TWO (Start a new Answer Booklet)

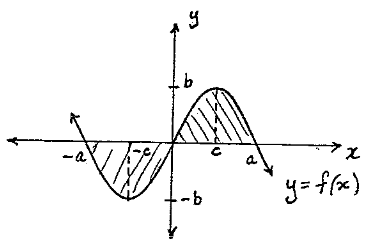
(a) Find the perpendicular distance from the point $(2,1)$ to the line $12x - 5y + 5 = 0$.

(b) Find the values of a and b, given that $(3\sqrt{2} - 1)^2 = a - b\sqrt{2}$.

(c) The limiting sum of the geometric series $a + \frac{a}{2} + \frac{a}{4} + \cdots$ is 6. Find the value of a.

(d) At any point on a curve, $\frac{dy}{dx} = 2x + 5$. If the curve passes through the point $P(1,-2)$, find the equation of the curve.

(e)



The diagram above shows a sketch of $y = f(x)$ with a maximum turning point at (c, b), a minimum turning point at $(-c, -b)$, and a point of inflexion at the origin.

(i) $y = f(x)$ is known to be an odd function. How does the graph illustrate this property?

(ii) For what values of x is $f'(x) < 0$?

(iii) Write down the co-ordinates of the point where the gradient of the curve is a maximum.

(iv) Find the value of $\int_{-a}^{a} f(x)\,dx$.

(f) Solve $\frac{1}{x+1} > 2$.

QUESTION THREE (Start a new Answer Booklet)

(a) Write down the equation of the locus of the point $P(x,y)$ which moves so that its distance from the point $C(2,-1)$ is always 2 units.

(b) Differentiate the following with respect to x:

(i) $y = 3x^4 + 2x^2 - 3$,

(ii) $y = \frac{3}{x}$,

(iii) $y = 6\sqrt{x}$,

(iv) $y = (4x^2 - 3x)^{10}$.

(c) Consider the function $f(x) = x(x-3)^2$.

(i) Find the intercepts with the x and y axes of the curve $y = f(x)$.

(ii) Show that $f'(x) = 3(x-1)(x-3)$, and find $f''(x)$.

(iii) Find the co-ordinates of any stationary points and determine their nature.

(iv) Find the co-ordinates of any points of inflexion.

(v) Sketch the graph, indicating all the important features.

QUESTION FOUR (Start a new Answer Booklet)

(a) Find each of the following indefinite integrals:

(i) $\int (5x^2 + 2x - 3)\,dx$,

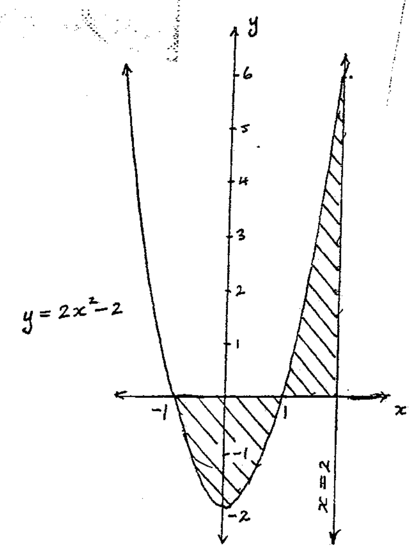
(ii) $\int \frac{x^3 - x}{x^3}\,dx$.

(b) (i) Copy and complete the table below for the function $y = 2^{-x}$.

x	-2	-1	0	1	2
y					

(ii) Use Simpson's rule with the five function values from the table to find an approximation for $\int_{-2}^{2} 2^{-x}\,dx$.

(c)



The diagram above shows the region bounded by the parabola $y = 2x^2 - 2$, the x-axis and the line $x = 2$. Calculate the area of this region.

(d) Consider the parabola with the equation $x^2 - 6x + 4y + 5 = 0$.

(i) Express the parabola in the form $(x-h)^2 = -4a(y-k)$, and hence find the co-ordinates of the vertex of the parabola.

(ii) Find the co-ordinates of the focus of the parabola.

(e) (i) Differentiate $y = \sqrt{3x^2 - 2x}$.

(ii) Hence find $\displaystyle\int \frac{3x-1}{\sqrt{3x^2-2x}}\,dx$.

QUESTION FIVE (Start a new Answer Booklet)

(a) (i) Complete the square to find the minimum value of the quadratic function $y = x^2 + 4x - 6$.

(ii) Find the values of k for which the quadratic equation $x^2 + 4x - (6+k) = 0$ has real roots.

(iii) Let the solutions of the quadratic equation $x^2 + 4x - 6 = 0$ be α and β. Find :

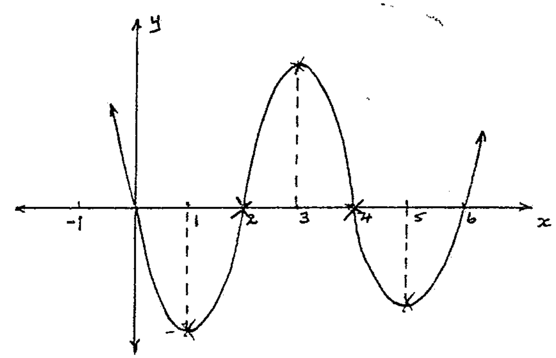
(α) $\alpha + \beta$,

(β) $\alpha\beta$,

(γ) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$.

(b) Solve $2\sin^2\theta - 5\sin\theta + 2 = 0$, for $-180^\circ \le \theta \le 180^\circ$.

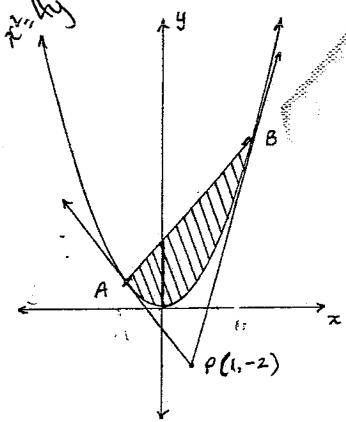
(c)



The diagram above shows the curve $y = f(x)$ with points of inflexion at $(2, 0)$ and $(4, 0)$. Sketch the graph of $y = f'(x)$.

QUESTION SIX (Start a new Answer Booklet)

(a)



The diagram above shows the parabola $x^2 = 4y$ and the chord of contact AB from the external point $P(1,-2)$.

(i) Write down the equation of the chord of contact.

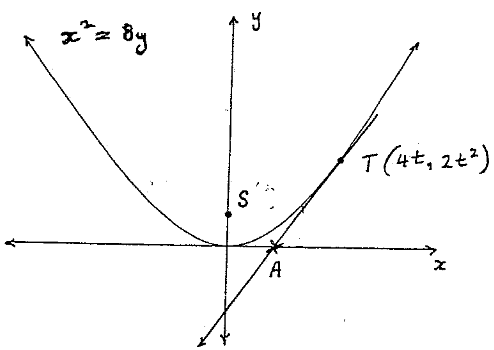
(ii) Show that the endpoints of the chord are $A(-2,1)$ and $B(4,4)$.

(iii) Find the area of the shaded region.

(b) (i) Sketch $y = |x-4|$.

(ii) For what values of c does the equation $|x-4| = \frac{1}{2}x + c$ have two distinct solutions?

(c)



The diagram above shows the parabola $x^2 = 8y$ with focus $S(0,2)$. The tangent at the variable point $T(4t, 2t^2)$ meets the x-axis at A.

(i) Show that the equation of the tangent at T is $y = tx - 2t^2$.

(ii) Find the co-ordinates of the mid-point M of TA.

(iii) Find the Cartesian equation of the locus of M.

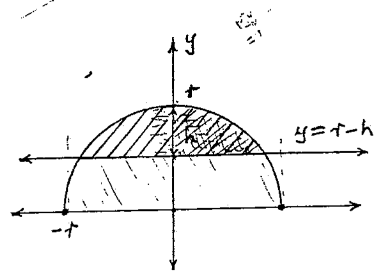
QUESTION SEVEN (Start a new Answer Booklet)

(a) Use mathematical induction to prove that

$$\frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{(n+1)(n+2)} = \frac{n}{2(n+2)}$$

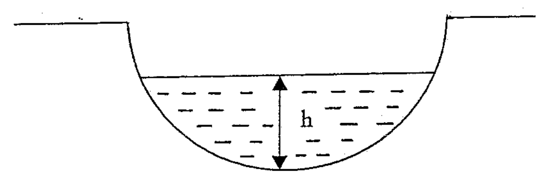
for all positive integers n.

(b)



The diagram above shows the region bounded by the semi-circular curve $y = \sqrt{r^2 - x^2}$ and the line $y = r - h$, where $0 \leq h \leq r$. Show that the volume of the solid formed when this region is rotated about the y-axis is given by $V = \frac{\pi}{3}h^2(3r - h)$.

(c)



Water is pumped into a hemispherical dam of radius 4 metres at a rate of 3π cubic metres per hour. How fast is the water rising when the water depth h is 3 metres? (Hint: You will need to use the formula derived in part (b).)

(d) Draw a possible sketch of a curve $y = f(x)$ for which:

- the domain is the set of all real numbers, and
- the curve is differentiable for all values of x except $x = 2$, and
- the x-axis is a horizontal asymptote in both directions, and
- the function has positive values for $x > 0$, and negative values for $x < 0$.

QUESTION EIGHT (Start a new Answer Booklet)

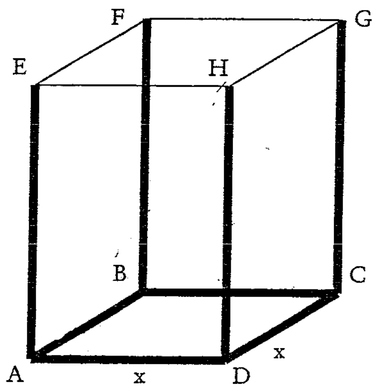
(a) Find where the curve $y = \dfrac{1-x^2}{1+x^2}$ is concave down.

(b) (i) Prove that $\dfrac{1}{(1+nx)(1+(n+1)x)} = \dfrac{1}{x}\left(\dfrac{1}{1+nx} - \dfrac{1}{1+(n+1)x}\right)$.

(ii) Hence find the sum of the first n terms of the series

$$\frac{1}{(1+x)(1+2x)} + \frac{1}{(1+2x)(1+3x)} + \frac{1}{(1+3x)(1+4x)} + \cdots$$

(c)



The diagram above shows a container of fixed volume V with a square base of length x units. The eight edges AB, BC, CD, DA, AE, BF, CG and DH are constructed of metal piping for extra strength.

(i) Show that the total length L of these eight edges is $L = 4x + \dfrac{4V}{x^2}$.

(ii) Find the minimum value of L.

(d) (i) Factorise $a^5 - b^5$.

(ii) Show that $77^5 - 22^5 + 32^4 - 12^4$ is divisible by 5. You may not use your calculator.

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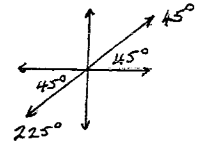
FORM V 3 UNIT 2001 SGS YEARLY SOLUTIONS.

QUESTION 1

(a) $\tan\theta = 1$

$\theta = 45°$ or $\theta = 225°$ ✓ ✓



(b) $\dfrac{x^3-8}{x^2-4} = \dfrac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}$

$= \dfrac{x^2+2x+4}{x+2}$ ✓

c) (i) Put $a + (n-1)d = 58$ ✓

$-2 + (n-1)3 = 58$

$3n - 5 = 58$

$3n = 63$

$n = 21$ ✓

58 is the 21st term.

(ii) $S_n = \frac{n}{2}(a+l)$

$S_{21} = \frac{21}{2} \times 56$

$S_{21} = 588$ ✓

d) $|x+1| < 2$

$x+1 < 2$ and $x+1 > -2$

$x < 1$ and $x > -3$

$-3 < x < 1$ ✓

e) $\dfrac{x}{\sin 30°} = \dfrac{8}{\sin 45°}$ ✓

$x = \dfrac{8\sin 30°}{\sin 45°}$

$x = 4 \times \frac{1}{2} \times \frac{\sqrt{2}}{1}$

$x = 4\sqrt{2}$ cm ✓

(f)(i) gradient $= \dfrac{y_2-y_1}{x_2-x_1}$

$= \dfrac{4-0}{2-1}$

$= 4$ ✓

(ii) $y = 2x^2 - 2x$

$\dfrac{dy}{dx} = 4x - 2$ ✓ At $B(2,4)$,

gradient $= 6$. ✓

(g)(i) $f(x) = x^2 + 3x$

$f(x+h) = (x+h)^2 + 3(x+h)$

$= x^2 + 2xh + h^2 + 3x + 3h$

$f(x+h) - f(x) = 2xh + h^2 + 3h$ ✓

(ii) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{x^2+2xh+h^2+3x+3h-x^2-}{h}$

$= \lim\limits_{h\to 0} \dfrac{2xh + h^2 + 3h}{h}$ ✓

$= \lim\limits_{h\to 0} \dfrac{h(2x+3+h)}{h}$

$= \lim\limits_{h\to 0} (2x+3+h)$ ✓

$= 2x + 3$.

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QUESTION 2

a) $12x - 5y + 5 = 0 \qquad (2,1)$

$$d = \left| \frac{ax_0 + by_0 + c}{\sqrt{a^2+b^2}} \right| \checkmark$$

$$= \frac{24 - 5 + 5}{\sqrt{144+5}}$$

$$= \frac{24}{13} \text{ units} \checkmark$$

b) $(3\sqrt{2} - 1)^2 = a - b\sqrt{2}$

$18 - 6\sqrt{2} + 1 = a - b\sqrt{2}$

$19 - 6\sqrt{2} = a - b\sqrt{2} \checkmark$

$a = 19$ and $b = 6 \checkmark$

) $a + \frac{a}{2} + \frac{a}{4} + \cdots$, is a GP with $r = \frac{1}{2}$.

Put $S_\infty = 6$

$\frac{a}{1-r} = 6 \checkmark$

$a = 6 - 6r$

$a = 6 - 3$

$a = 3 \checkmark$

) $\frac{dy}{dx} = 2x + 5$

$y = x^2 + 5x + c \checkmark$

Since the curve passes through $P(1,-2)$,

$-2 = 1 + 5 + c$

$c = -8$

So the equation of the curve is $y = x^2 + 5x - 8 \checkmark$

(e)(i) The graph has point symmetry about the origin. (or rotationa[l] symmetry of order 2) $\checkmark$

(ii) When $f'(x) < 0$, the curve is [...] decreasing. Hence $x < -c$ or $x > c$

(iii) The gradient of the curve is a maximum at the point of inflexion $(0,0)$. $\checkmark$

(iv) $\int_{-a}^{a} f(x)\,dx = 0$, since the function is odd. $\checkmark$

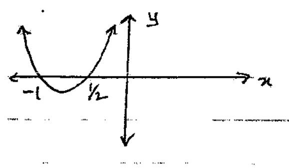
(f) $\frac{1}{x+1} > 2$ multiplying by $(x+1)^2$,

$x+1 > 2(x+1)^2$

$x+1 > 2(x^2+2x+1)$

$2x^2 + 3x + 1 < 0 \checkmark$

$(2x+1)(x+1) < 0$



$\checkmark$

$-1 < x < -\frac{1}{2} \checkmark$

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QUESTION 3 The locus is a

(a) Circle, centre $(2,-1)$, radius $= 2$

$(x-2)^2 + (y+1)^2 = 4 \checkmark$

b) (i) $y = 3x^4 + 2x^2 - 3$

$\frac{dy}{dx} = 12x^3 + 4x \checkmark$

(ii) $y = 3x^{-1}$

$\frac{dy}{dx} = -3x^{-2}$

$= \frac{-3}{x^2} \checkmark$

(iii) $y = 6x^{1/2}$

$\frac{dy}{dx} = 3x^{-1/2}$

$= \frac{3}{\sqrt{x}} \checkmark$

(iv) $y = (4x^2 - 3x)^{10}$

$\frac{dy}{dx} = 10(4x^2-3x)^9(8x-3) \checkmark$

) $f(x) = x(x-3)^2$

(i) Y intercept: $(0,0) \checkmark$

X intercepts: $(0,0)$ and $(3,0) \checkmark$

(ii) $f(x) = x(x-3)^2$

$f'(x) = (x-3)^2 + 2x(x-3)$

$= (x-3)\{(x-3) + 2x\} \checkmark$

$= (x-3)(3x-3)$

$= 3(x-3)(x-1)$

$f'(x) = 3x^2 - 12x + 9$

$f''(x) = 6x - 12 \checkmark$

(iii) For stationary points;

put $f'(x) = 0$

$3(x-3)(x-1) = 0$

$x = 1$ or $x = 3$. Hence there are stationa[ry] points at $(1,4)$ and $(3,0)$. $\checkmark$

At $(1,4)$, $f''(1) = -6$

$f''(1) < 0$, so there is a local maximum at $(1,4)$. $\checkmark$

At $(3,0)$, $f''(3) = 6$

$f''(3) > 0$, so there is a local minimum at $(3,0)$. $\checkmark$

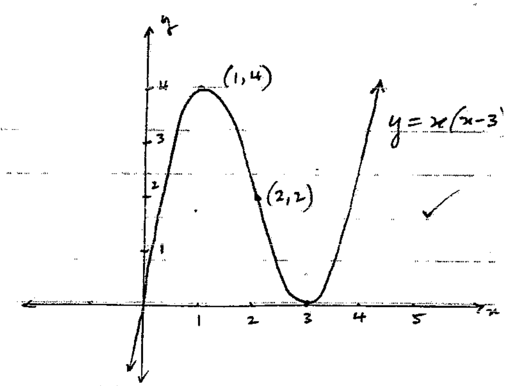
(iv) Put $f''(x) = 0$

$6x - 12 = 0$

$x = 2$, so there is a possibl[e] point of inflexion at $(2,2)$. $\checkmark$

x	1.9	2	2.1
$f''(x)$	$-$	0	$+$

A change in concavity occurs at $(2,2)$, so there is a point of inflexion at $(2,2)$ $\checkmark$



$\checkmark$

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QUESTION 4

(a)(i) $\int (5x^2+2x-3)\,dx$

$= \frac{5}{3}x^3 + x^2 - 3x + c$ ✓

$\int (1-x^{-2})\,dx = x + \frac{1}{x} + c$ ✓ ✓

(b) (i) $y = 2^{-x}$

-2	-1	0	1	2
4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

✓

(ii) Simpson's Rule.

$\int_{-2}^{2} 2^{-x}\,dx = \int_{-2}^{0} 2^{-x}\,dx + \int_{0}^{2} 2^{-x}\,dx$

$\doteqdot \frac{1}{3}(4+4\times2+1) + \frac{1}{3}(1+4\times\frac{1}{2}+\frac{1}{4})$ ✓

$\doteqdot \frac{13}{3} + \frac{13}{12}$

$\doteqdot \frac{65}{12}$

$\doteqdot 5\frac{5}{12}$ ✓

(c) Area $= -\int_{-1}^{1}(2x^2-2)\,dx + \int_{1}^{2}(2x^2-2)\,dx$ ✓

$= -\left[\frac{2x^3}{3} - 2x\right]_{-1}^{1} + \left[\frac{2}{3}x^3 - 2x\right]_{1}^{2}$ ✓

$= -\left[\left(\frac{2}{3}-2\right) - \left(-\frac{2}{3}+2\right)\right] + \left[\left(\frac{16}{3}-4\right) - \left(\frac{2}{3}-2\right)\right]$

$= \frac{8}{3} + \frac{8}{3}$

$= 5\frac{1}{3}$ square units. ✓

(d) (i) $x^2 - 6x + 4y + 5 = 0$

$(x-3)^2 - 9 + 4y + 5 = 0$

$(x-3)^2 = -4y + 4$

Hence the vertex is $(3,1)$ and focal length $= 1$. ✓

(ii) the focus is $(3,0)$. ✓

(e)

(i) $y = (3x^2-2x)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(3x^2-2x)^{-\frac{1}{2}}(6x-2)$ ✓

$= \frac{3x-1}{\sqrt{3x^2-2x}}$ ✓

(ii) From part (i),

$\int \frac{3x-1}{\sqrt{3x^2-2x}}\,dx = \sqrt{3x^2-2x} + c$ ✓

QUESTION 5

(a) (i) $y = x^2 + 4x - 6$

$y = (x+2)^2 - 4 - 6$

$y = (x+2)^2 - 10$ ✓

minimum value $= -10$. ✓

(ii) $x^2 + 4x - (6+k) = 0$

For real roots $\Delta \geq 0$.

$\Delta = b^2 - 4ac$

$= 16 + 4(6+k)$

$= 40 + 4k$. ✓

$40 + 4k \geq 0$

$4k \geq -40$

For real roots $k \geq -10$. ✓

(iii) $x^2 + 4x - 6 = 0$

$\alpha)$ $\alpha + \beta = -\frac{b}{a}$

$= -4$ ✓

$\beta)$ $\alpha\beta = \frac{c}{a}$

$= -6$ ✓

$\gamma)$ $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2+\alpha^2}{\alpha^2\beta^2}$

$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$ ✓

$= \frac{16+12}{36}$

$= \frac{7}{9}$ ✓

b) $2\sin^2\theta - 5\sin\theta + 2 = 0$

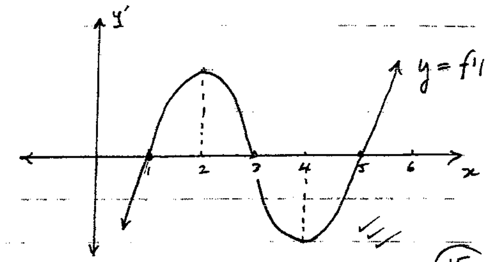
$(2\sin\theta - 1)(\sin\theta - 2) = 0$ ✓

$2\sin\theta = 1$ or $\sin\theta = 2$ ✓

$\sin\theta = \frac{1}{2}$ ✓ no solutions.

$\theta = 30°$ or $\theta = 150°$ ✓

(c)



✓✓✓

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Question 6

a)(i) The chord of contact from $P(1,-2)$ is

$xx_0 = 2a(y+y_0)$. When

$a=1$, $x = 2(y-2)$

$x = 2y-4$ ✓

$x-2y+4=0$

(ii) Solve $x^2 = 4ay$ ...... ①

$x = 2y-4$ ..... ②

Simultaneously.

②^2 $(2y-4)^2 = 4ay$

$4y^2-16y+16 = 4y$

$4y^2-20y+16=0$

$y^2-5y+4=0$ ✓

$(y-1)(y-4)=0$

When $y=1$, $x=-2$. When $y=4$, $x=4$. So the endpoints are $A(-2,1)$ and $B(4,4)$. ✓

iii) $\text{Area} = \int_{-2}^{4} \left(\frac{1}{2}x+2-\frac{1}{4}x^2\right)dx$ ✓

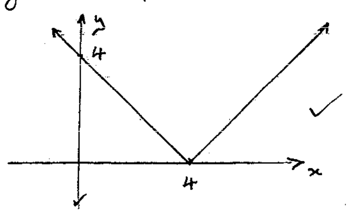
$= \left[\frac{1}{4}x^2+2x-\frac{1}{12}x^3\right]_{-2}^{4}$ ✓

$=\left(4+8-\frac{64}{12}\right)-\left(1-4+\frac{8}{12}\right)$

$= 15-\frac{72}{12}$

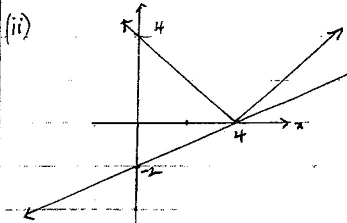
$= 9$ square units. ✓

b) (i) $y = |x-4|$



✓

(ii)



✓

$|x-4| = \frac{1}{2}x+c$ will have two distinct solutions when $c > -2$. ✓

(c)(i) $y = \frac{1}{8}x^2$

$\frac{dy}{dx} = \frac{1}{4}x$ ✓

the gradient is t. So the tangent.

T is $y-y_1 = m(x-x_1)$

$y-2t^2 = t(x-4t)$ ✓

$y-2t^2 = tx-4t^2$

$y = tx-2t^2$

(ii) Put $y=0$

$tx = 2t^2$

$x = 2t$. Hence $A = (2t, 0)$

So the mid-point M is $(3t, t^2)$. ✓

(iii) $x = 3t$ .... ①

$y = t^2$ ..... ②

From ① $t = \frac{x}{3}$ substituting into ② ✓

$y = \frac{x^2}{9}$

$x^2 = 9y$ ✓ This is a parabola with vertex $(0,0)$ and focal length $\frac{9}{4}$. 15

Question 7

a) To prove

$$\frac{1}{2\times3}+\frac{1}{3\times4}+\dots+\frac{1}{(n+1)(n+2)} = \frac{n}{2(n+2)}$$

A// When $n=1$, $RHS = \frac{1}{2\times3}$

$= \frac{1}{6}$

$LHS = \frac{1}{2\times3}$ $= LHS$. ✓

$= \frac{1}{6}$ The statement is true for $n=1$.

B// Suppose k is a positive integer for which the statement is true.

that is $\frac{1}{2\times3}+\frac{1}{3\times4}+\dots\frac{1}{(k+1)(k+2)} = \frac{k}{2(k+2)}$ ✓

We prove the statement true for $n=k+1$. That is we prove ✓

$$\frac{1}{2\times3}+\frac{1}{3\times4}+\dots+\frac{1}{(k+1)(k+2)}+\frac{1}{(k+2)(k+3)} = \frac{k+1}{2(k+3)}$$

$LHS = \frac{1}{2\times3}+\frac{1}{3\times4}+\dots+\frac{1}{(k+1)(k+2)}+\frac{1}{(k+2)(k+3)}$

$= \frac{k}{2(k+2)}+\frac{1}{(k+2)(k+3)}$ (by the induction hypothesis.)

$= \frac{k(k+3)+2}{2(k+2)(k+3)}$ ✓

$= \frac{k^2+3k+2}{2(k+2)(k+3)}$

$= \frac{(k+1)(k+2)}{2(k+2)(k+3)}$

$= \frac{k+1}{2(k+3)}$

$= RHS$.

C// It follows from parts A and B by Mathematical induction that the ✓ statement is true for all positive integers n.

(b) $V = \pi\int_{r-h}^{r} r^2-x^2 \, dy$ ✓

$= \pi\left[r^2x-\frac{x^3}{3}\right]_{r-h}^{r}$ ✓

$= \pi\left[\left(r^3-\frac{r^3}{3}\right)-\left(r^2(r-h)-\frac{(r-h)^3}{3}\right)\right]$

$= \pi\left[\frac{2r^3}{3}-\left(r^3-r^2h-\frac{1}{3}(r^3-3r^2h+3rh^2\right.\right.$ ✓

$= \pi\left[\frac{2}{3}r^3-\left(\frac{2}{3}r^3-rh^2-\frac{1}{3}h^3\right]\right.$

$= \pi\left[rh^2-\frac{1}{3}h^3\right]$

$V = \frac{\pi}{3}h^2(3r-h)$. ✓

(c) $V = \frac{\pi}{3}h^2(3r-h)$

$= 4\pi h^2-\frac{\pi}{3}h^3$, since $r=4$.

$\frac{dV}{dt} = \frac{dV}{dh}\times\frac{dh}{dt}$ (chain rule)

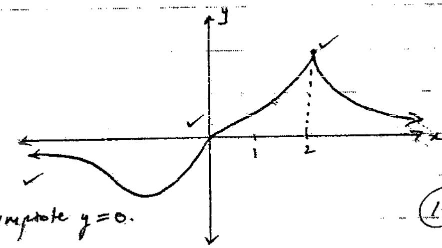
$\frac{dV}{dt} = (8\pi h-\pi h^2)\frac{dh}{dt}$. ✓

Substitute $\frac{dV}{dt} = 3\pi$ and $h=3$.

$3\pi = (24\pi-9\pi)\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{5}$ m/hr. ✓

(d)



asymptote $y=0$.

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QUESTION 8

(a) $y = \dfrac{1-x^2}{1+x^2}$

$\dfrac{dy}{dx} = \dfrac{(1+x^2)(-2x)-(1-x^2)2x}{(1+x^2)^2}$

$= \dfrac{-2x(1+x^2+1-x^2)}{(1+x^2)^2}$

$\dfrac{dy}{dx} = \dfrac{-4x}{(1+x^2)^2}$ ✓

$\dfrac{d^2y}{dx^2} = \dfrac{(1+x^2)^2(-4)-(-4x)2(1+x^2)2x}{(1+x^2)^4}$

$= \dfrac{-4(1+x^2)\{1+x^2-4x^2\}}{(1+x^2)^4}$

$= \dfrac{4(3x^2-1)}{(1+x^2)^3}$ ✓

So y'' has zeros at $x = \frac{1}{\sqrt{3}}$ and $x = -\frac{1}{\sqrt{3}}$ and no discontinuities.

x	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1
$\frac{d^2y}{dx^2}$	1	0	-4	0	1

The curve is concave down for $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$. ✓

b) (i) $RHS = \dfrac{1}{x}\left(\dfrac{1}{1+nx} - \dfrac{1}{1+(n+1)x}\right)$

$= \dfrac{1}{x}\left(\dfrac{1+(n+1)x-(1+nx)}{(1+nx)(1+(n+1)x)}\right)$

✓ $= \dfrac{1}{x}\left(\dfrac{1+nx+x-1-nx}{(1+nx)(1+(n+1)x)}\right)$

$= \dfrac{1}{x}\left(\dfrac{nx}{(1+nx)(1+(n+1)x)}\right)$

$= \dfrac{1}{(1+nx)(1+(n+1)x)}$

$= LHS.$

(ii)

$\dfrac{1}{(1+x)(1+2x)} + \dfrac{1}{(1+2x)(1+3x)} + \dfrac{1}{(1+3x)(1+4x)} + \dots + \dfrac{1}{(1+nx)(1+(n+1)x)}$ using the result from part (i)

$= \dfrac{1}{x}\left(\dfrac{1}{1+x} - \dfrac{1}{1+2x} + \dfrac{1}{1+2x} - \dfrac{1}{1+3x} + \dots + \dfrac{1}{1+nx} - \dfrac{1}{1+(n+1)x}\right)$ ✓

$= \dfrac{1}{x}\left(\dfrac{1}{1+x} - \dfrac{1}{1+(n+1)x}\right)$ ✓

$= \dfrac{1}{x}\left(\dfrac{1+(n+1)x-(1+x)}{(1+x)(1+(n+1)x)}\right)$

$= \dfrac{1}{x} \times \dfrac{nx}{(1+x)(1+(n+1)x)}$

$= \dfrac{n}{(1+x)(1+(n+1)x)}$ ✓

(c)(i) let $HD = y$.

Since $V = x^2y$,

$y = \dfrac{V}{x^2}$.

$L = 4x + 4y$

$L = 4x + \dfrac{4V}{x^2}$ ✓

(ii) $\dfrac{dL}{dx} = 4 - 8Vx^{-3}$

$4 - \dfrac{8V}{x^3} = 0$

$4x^3 - 8V = 0$

$4x^3 = 8V$

$x = (2V)^{\frac{1}{3}}$ ✓

So $\dfrac{dL}{dx}$ has a zero at $x = (2V)^{\frac{1}{3}}$.

$\dfrac{d^2L}{dx^2} = 24Vx^{-4}$ ✓

So $\dfrac{d^2L}{dx^2} > 0$ for all x, so L has a minimum at $x = (2V)^{\frac{1}{3}}$.

ii) When $x = (2V)^{\frac{1}{3}}$,

$L = 4x + \dfrac{4V}{x^2}$

$L = 4\left\{(2V)^{\frac{1}{3}} + \dfrac{V}{(2V)^{\frac{2}{3}}}\right\}$ ✓

$= 4\left\{(2V)^{\frac{1}{3}} + 2^{-\frac{2}{3}}.V^{\frac{1}{3}}\right\}$

$= 2^2 \times 2^{\frac{1}{3}}V^{\frac{1}{3}} + 2^2 \times 2^{-\frac{2}{3}} \times V^{\frac{1}{3}}$

$= 4(2V)^{\frac{1}{3}} + 2^{\frac{4}{3}}.V^{\frac{1}{3}}$

$= 4(2V)^{\frac{1}{3}} + 2.2^{\frac{1}{3}}V^{\frac{1}{3}}$

$= 6(2V)^{\frac{1}{3}}$ ✓

which is the minimum value of L.

d)

(i) $a^5 - b^5 = (a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)$ ✓

(ii) $77^5 - 22^5 + 32^4 - 12^4$

$= (77-22)m + (32-12)n$ ✓

$= 55m + 20n$ (where m and n are positive integers.)

$= 5(11m+4n)$ ✓

The number is divisible by 5.

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