SYDNEY GRAMMAR SCHOOL HALF-YEARLY EXAMINATION 2003

FORM V MATHEMATICS & EXTENSION 1

Time allowed: 2 hours **Exam date:** 14th May 2003

Instructions:

All questions may be attempted.
All questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.

Collection:

The writing booklets will be collected in one bundle.
Start each question in a new writing booklet.
If you use a second booklet for a question, place it inside the first. <u>Don't staple</u>.

Checklist:

Folded A3 writing booklets required — 8 per boy.
Candidature: 136 boys

<u>QUESTION ONE</u> (Start a new writing booklet)

(a) Given that $f(x) = x^2 - 4x - 2$, find the value of $f(-1)$.

(b) Simplify $\sqrt{48}$.

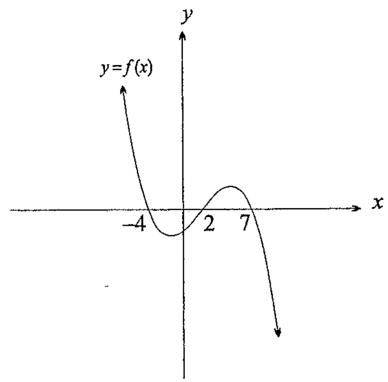
(c) Factorise:

(i) $1 + x^3$

(ii) $2x^2 - 3x + 1$

(d) If N is a negative number, write down the value of $|N|$.

(e)



For what values of x in the diagram above is $f(x) > 0$?

(f) Given the arithmetic sequence 3, 10, 17, ..., find:

(i) the 1001st term,

(ii) the sum of the first 1001 terms.

(g) Calculate the perpendicular distance from the point $(1, -2)$ to the line $3x - 4y + 5 = 0$.

(h) Write $0{\cdot}\dot{1}\dot{5}$ as a fraction in lowest terms. Show full working.

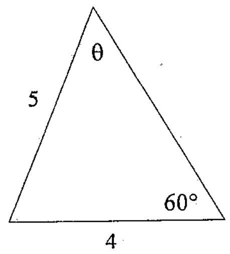
Exam continues next page ...

QUESTION TWO (Start a new writing booklet)

(a) Write the fraction $\dfrac{1}{2\sqrt{5}-3}$ with a rational denominator.

(b) (i) Write down the exact value of $\sin 60°$.

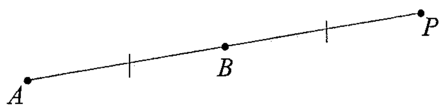
(ii)



Use the sine rule in the triangle drawn above to find the exact value of $\sin\theta$.

(c) The limit of the sum of an infinite geometric series is $\frac{14}{15}$, and the first term is $\frac{1}{3}$. Find the common ratio.

(d)



In the diagram above, $AB = BP$.
In what ratio does the point P divide the interval AB?

(e) Determine the gradient of a line that is perpendicular to the line $8x + 3y - 5 = 0$.

(f) Consider the function $f(x) = 2x - 5$.

(i) Find $f^{-1}(x)$.

(ii) Show that $f^{-1}(f(x)) = x$.

Exam continues overleaf ...

QUESTION THREE (Start a new writing booklet)

(a) Solve for x:

(i) $1 - 3x > 16$

(ii) $|x + 7| < 6$

(iii) $x^2 + 13x > 90$

(iv) $\dfrac{x+1}{x} \le 0$

(b) The function $f(x)$ is odd, and it is known that $f(3) = 7$.

(i) What is the value of $f(-3)$?

(ii) Sketch a possible graph of $y = f(x)$.

(c) (i) Sketch the graph of the function $y = -\sqrt{2 - x^2}$.

(ii) Hence state the range of $y = -\sqrt{2 - x^2}$.

QUESTION FOUR (Start a new writing booklet)

(a) (i) Find the equation of the axis of symmetry of the parabola $y = 2 + 5x - x^2$.

(ii) Hence find the coordinates of the vertex of the parabola.

(b) Shade the region of the number plane where $-2 < x < 5$ and $-1 < y < 3$.

(c) A line has x-intercept k and y-intercept $2k$, where $k \neq 0$.

(i) Show that its equation is $2x + y = 2k$.

(ii) Sketch the line in the case where $k < 0$.

(d) At the start of the year 2000, John started working as an actuary . In 2000 his income was \$33 696, in 2001 his income was \$50 544 and in 2002 his income was \$75 816.

(i) Show that his first three yearly incomes are in geometric progression.

(ii) By what percentage has John's income increased each year?

(iii) Assuming that John's income continues to increase as it has over the first three years, calculate, correct to the nearest dollar, John's total income over his first 15 years as an actuary.

(iv) Show that John's income in dollars in his nth year as an actuary will be

$$13 \times 2^{6-n} \times 3^{n+3}.$$

Exam continues next page ...

QUESTION FIVE (Start a new writing booklet)

(a) Evaluate $\sec^2 81°47'$ correct to two significant figures.

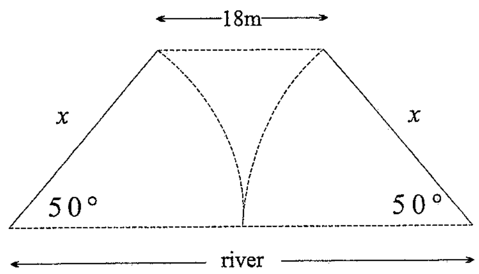
(b) Solve for x over the domain $0° \le x \le 360°$, giving solutions correct to the nearest minute where necessary.

(i) $\tan x = -\dfrac{1}{\sqrt{3}}$

(ii) $1 + \tan x = 2\cot x$

(c) Prove the identity $\dfrac{\sin\theta}{\sec\theta - \cos\theta} = \cot\theta$.

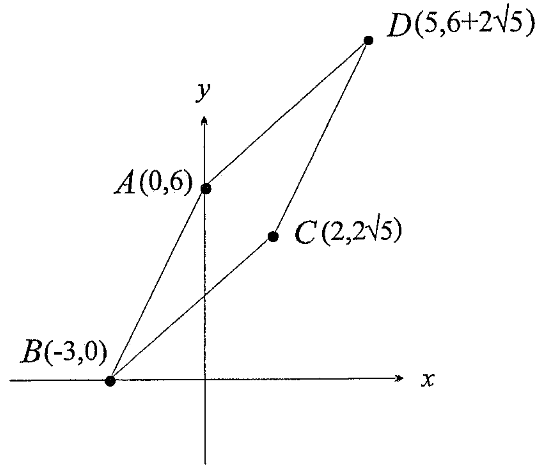
(d)



A bridge spans a river, and the two identical sections of the bridge, each of length x metres, can be raised to allow tall boats to pass. When the two sections are fully raised, they are each inclined at $50°$ to the horizontal, and there is an 18 metre gap between them, as shown in the diagram above. Calculate the width of the river in metres, correct to one decimal place.

QUESTION SIX (Start a new writing booklet)

(a)



The points $A(0,6)$, $B(-3,0)$, $C(2,2\sqrt{5})$ and $D(5,6+2\sqrt{5})$, as shown in the diagram above, are the vertices of a quadrilateral $ABCD$.

(i) Find the coordinates of the midpoint of the diagonal AC.

(ii) Find the coordinates of the midpoint of the diagonal BD.

(iii) Based on your results in parts (i) and (ii), what type of special quadrilateral is $ABCD$?

(iv) Show that the diagonals AC and BD are perpendicular.

(v) What can now be said about quadrilateral $ABCD$?

(b) (i) Show that the line $3x + 17y - 11 + k(13x - 5y + 7) = 0$ has gradient $\dfrac{13k+3}{5k-17}$.

(ii) Hence, or otherwise, find the equation of the vertical line that passes through the point of intersection of the lines $3x + 17y - 11 = 0$ and $13x - 5y + 7 = 0$.

(c) (i) Describe a pair of transformations by which the graph of a function $y = f(x)$ is transformed to the graph of $y = f(4 - x)$.

(ii) Describe a single transformation by which the graph of a function $y = f(x)$ is transformed to the graph of $y = f(4 - x)$.

QUESTION SEVEN (Start a new writing booklet)

(a) Consider the curve $y = \dfrac{x^2+x-6}{x^2-x-6}$.

(i) Find the y-intercept.

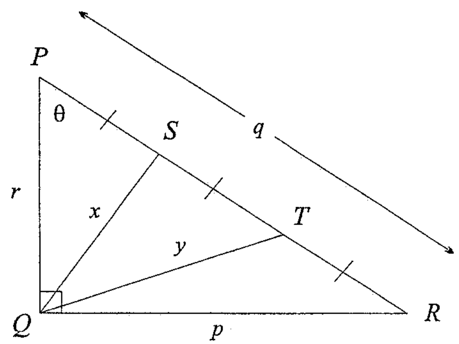
(ii) Find the x-intercepts.

(iii) Write down the equations of the vertical asymptotes.

(iv) Find the equation of the horizontal asymptote.

(v) Sketch the curve, showing all the above features.

(b)



The right-angled triangle PQR has its hypotenuse PR trisected at the points S and T, as shown in the diagram above.

Let $QR = p$, $PR = q$, $PQ = r$, $QS = x$ and $QT = y$. Also let $\angle QPR = \theta$.

(i) Show that $\cos\theta = \dfrac{9r^2+q^2-9x^2}{6qr}$.

(ii) Use $\triangle QRT$ to show that $\sin\theta = \dfrac{9p^2+q^2-9y^2}{6pq}$.

(iii) Deduce that $5q^2 = 9(x^2+y^2)$.

PLEASE TURN OVER FOR QUESTION 8

QUESTION EIGHT (Start a new writing booklet)

(a) An arithmetic series has first term a and common difference d.
The sum of the first $4n$ terms is twenty times the sum of the first n terms.

(i) Show that $d = \dfrac{8a}{4-n}$.

(ii) Hence find all the arithmetic series with first term 1 and positive common difference that satisfy the above condition.

(b) Find $\displaystyle\sum_{k=1}^{215} \frac{1}{\sqrt[3]{k^2}+\sqrt[3]{k(k+1)}+\sqrt[3]{(k+1)^2}}$.

DS

SOLUTIONS TO F5 3U MIDYEAR EXAM (2003)
(Total is 120.)

(1) (a) $f(-1) = (-1)^2 - 4(-1) - 2$
$= 1 + 4 - 2$
$= 3$ ✓

(b) $\sqrt{48} = 4\sqrt{3}$ ✓

(c)(i) $(1+x)(1-x+x^2)$ ✓

(ii) $(2x-1)(x-1)$ ✓

(d) $-N$ ✓

(e) $x < -4$ or $2 < x < 7$ ✓

No penalty if comma instead of "or". ("and" is wrong)

(f)(i) $T_{1001} = 3 + 1000 \times 7$
$= 7003$ ✓✓

(ii) $S_{1001} = \frac{1001}{2}(3 + 7003)$
$= 3\,506\,503$ ✓✓

(g) $d = \frac{|3(1) - 4(-2) + 5|}{\sqrt{3^2 + (-4)^2}}$ ✓
$= \frac{16}{5}$ or $3\frac{1}{5}$ units ✓

(h) Let $x = 0.\dot{1}\dot{5}$
$\therefore 100x = 15.\dot{1}\dot{5}$ ✓
$\therefore 99x = 15$ ✓
$\therefore 0.\dot{1}\dot{5} = \frac{15}{99}$
$= \frac{5}{33}$ ✓

Full marks for limiting sum of a geometric series. 2 marks only for $0.\dot{1}\dot{5} = \frac{15}{99} = \frac{5}{33}$, as full working is required.

(2)(a) $\frac{1}{2\sqrt{5}-3} \times \frac{2\sqrt{5}+3}{2\sqrt{5}+3}$ ✓ $= \frac{2\sqrt{5}+3}{11}$ ✓

(b)(i) $\sin 60° = \frac{\sqrt{3}}{2}$ ✓

(ii) $\frac{\sin\theta}{4} = \frac{\sin 60°}{5}$ ✓
$\therefore \sin\theta = 4 \times \frac{\sqrt{3}}{2} \times \frac{1}{5}$
$= \frac{2\sqrt{3}}{5}$ ✓

(c) $\frac{14}{15} = \frac{\frac{1}{3}}{1-r}$ ✓
$\therefore 14 - 14r = 5$ ✓
$14r = 9$
$r = \frac{9}{14}$ ✓

(d) $-2:1$ or $2:-1$ ✓✓

Award one mark for $2:1$ or $-1:2$ or $1:-2$.

(e) The given line has gradient $-\frac{8}{3}$, so a perpendicular line has gradient $\frac{3}{8}$. ✓✓

(f)(i) Let $y = 2x - 5$.
The inverse has equation
$x = 2y - 5$ ✓
i.e. $y = \frac{x+5}{2}$
So $f^{-1}(x) = \frac{x+5}{2}$. ✓

$y = \frac{x+5}{2}$ is enough to get the mark.

(ii) $f^{-1}(f(x)) = f^{-1}(2x-5)$
$= \frac{(2x-5)+5}{2}$
$= x$ ✓

(3)(a)(i) $-3x > 15$

$x < -5$

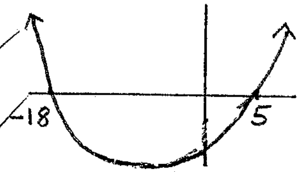
(ii) The distance between x and -7 is less than 6 units.

$\therefore -13 < x < -1$

(iii) $x^2 + 13x - 90 > 0$

$(x+18)(x-5) > 0$

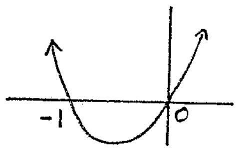
$x < -18$ or $x > 5$



(iv) Multiply both sides by x^2, where $x \neq 0$.

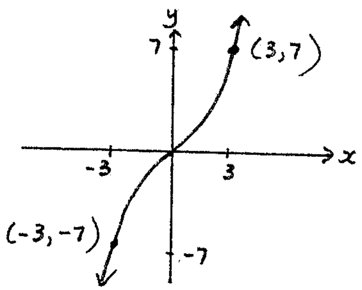
$\therefore x(x+1) \leq 0$

$-1 \leq x < 0$

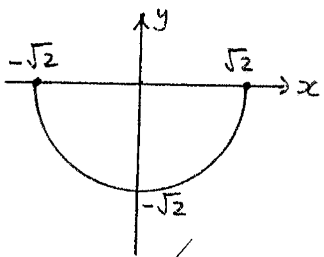


(b)(i) $f(-3) = -7$

(ii)



(c)(i)



Deduct one mark for: Upper semicircle, or intercepts of 2 instead of $\sqrt{2}$, or no intercepts mark

(ii) $-\sqrt{2} \leq y \leq 0$

(4)(a)(i) $x = \frac{-5}{2(-1)}$

$x = 2\frac{1}{2}$

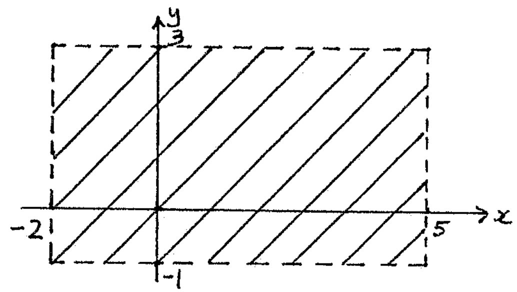
(ii) When $x = \frac{5}{2}$,

$y = 2 + \frac{25}{2} - \frac{25}{4}$

$= 8\frac{1}{4}$,

so the vertex is $(2\frac{1}{2}, 8\frac{1}{4})$.

(b)



Deduct one if boundaries full.

(c)(i) The line passes through $(k, 0)$ and $(0, 2k)$.

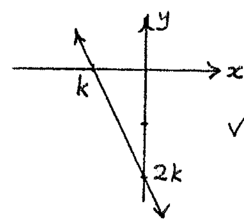
$\therefore m = \frac{2k-0}{0-k}$

$= -2$

$\therefore$ the line has equation $y = -2x + 2k$

i.e. $2x + y = 2k$

(ii)



No mark if intercepts marked $-k$ and $-2k$.

Award the mark (reluctantly) if no intercepts given.

(d)(i) $\frac{T_2}{T_1} = \frac{50544}{33696} = \frac{3}{2}$

and $\frac{T_3}{T_2} = \frac{75816}{50544} = \frac{3}{2}$,

so the sequence is geometric.

(ii) 50%

(iii) $S_{15} = \frac{33696(1 \cdot 5^{15} - 1)}{0 \cdot 5}$

$= \$29\,443\,153$

(iv)

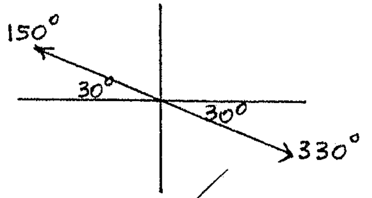
$T_n = 33696 \times (\frac{3}{2})^{n-1}$

$= 2^5 \times 3^4 \times 13 \times 3^{n-1} \times 2^{1-n}$

$= 13 \times 2^{6-n} \times 3^{n+3}$

(5) (a) 49 (Allow 48·96 or 49·0)

(b) (i)



$x = 150°$ or $330°$

(ii) $1 + \tan x = \dfrac{2}{\tan x}$

$\tan^2 x + \tan x - 2 = 0$

$(\tan x + 2)(\tan x - 1) = 0$

$\tan x = -2$ or $\tan x = 1$

$x = \underbrace{45°, 225°}, \underbrace{116°34' \text{ or } 296°34'}$

(c) $LHS = \dfrac{\sin\theta}{\frac{1}{\cos\theta} - \cos\theta} \times \dfrac{\cos\theta}{\cos\theta}$

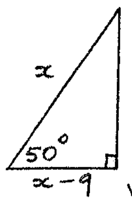
$= \dfrac{\sin\theta\cos\theta}{1 - \cos^2\theta}$

$= \dfrac{\sin\theta\cos\theta}{\sin^2\theta}$

$= \dfrac{\cos\theta}{\sin\theta}$

$= \cot\theta = RHS$

(d)



$\dfrac{x-9}{x} = \cos 50°$

$x - 9 = x\cos 50°$

$x(1 - \cos 50°) = 9$

$\therefore x = \dfrac{9}{1 - \cos 50°}$

Width of river $= 2x$

$= \dfrac{18}{1 - \cos 50°}$

$\doteqdot 50·4$ metres (No penalty if incorrect)

(6) (a) (i) $(1, 3 + \sqrt{5})$

(ii) $(1, 3 + \sqrt{5})$

(iii) The diagonals bisect each other, so it's a parallelogram.

(iv) $m_{AC} = \dfrac{2\sqrt{5} - 6}{2}$

$= \sqrt{5} - 3$

$m_{BD} = \dfrac{6 + 2\sqrt{5}}{8}$

$= \dfrac{3 + \sqrt{5}}{4}$

$\therefore m_{AC} \times m_{BD} = \frac{1}{4}(\sqrt{5} + 3)(\sqrt{5} - 3)$

$= \frac{1}{4}(5 - 9)$

$= -1$

$\therefore AC \perp BD$

(v) The diagonals bisect each other at right-angles, so it's a rhombus.

(b) (i) $3x + 17y - 11 + 13kx - 5ky + 7k = 0$

$(13k + 3)x + (17 - 5k)y + (7k - 11) = 0$

$(5k - 17)y = (13k + 3)x + (7k - 11)$

$y = \dfrac{13k + 3}{5k - 17}x + \dfrac{7k - 11}{5k - 17}$,

so $m = \dfrac{13k + 3}{5k - 17}$

(ii) A vertical line has undefined gradient, so $5k - 17 = 0$. i.e. $k = \frac{17}{5}$

So the equation of the line is

$3x + 17y - 11 + \frac{17}{5}(13x - 5y + 7) = 0$

$15x + \cancel{85y} - 55 + 221x - \cancel{85y} + 119 = 0$

$236x + 64 = 0$

$59x + 16 = 0$

(c) (i) Reflect in the y-axis, then shift 4 units right. (or shift 4 units left then reflect in y-axis) Deduct [illegible]

(ii) Reflect in $x = 2$.

(7)(a) $y = \dfrac{(x+3)(x-2)}{(x-3)(x+2)}$

(i) y-intercept is 1

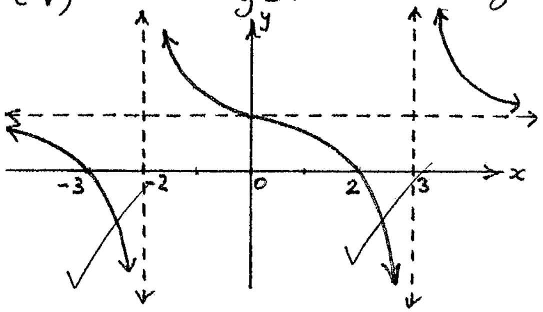
(ii) x-intercepts are -3 and 2

(iii) $x=-2$ and $x=3$

(iv) As $x \to \pm\infty$, $y \to 1$,

so $y=1$ is a horizontal asymptote.

(v)



(b)(i) In ΔPQS by the cosine rule,

$$\cos\theta = \frac{r^2+\left(\frac{q}{3}\right)^2 - x^2}{2\times r\times \frac{q}{3}} \times \frac{9}{9}$$

$$= \frac{9r^2+q^2-9x^2}{6qr} \quad (1)$$

(ii) In ΔQRT by the cosine rule,

$$\cos(90^\circ-\theta) = \frac{p^2+\left(\frac{q}{3}\right)^2 - y^2}{2\times p\times \frac{q}{3}} \times \frac{9}{9}$$

$$\therefore \sin\theta = \frac{9p^2+q^2-9y^2}{6pq} \quad (2)$$

(iii) In ΔPQR, $\cos\theta = \frac{r}{q}$ and $\sin\theta = \frac{p}{q}$.

$\therefore$ (1) becomes $3r^2 = 9x^2 - q^2$

and (2) becomes $3p^2 = 9y^2 - q^2$.

(1)+(2): $3(r^2+p^2) = 9x^2 + 9y^2 - 2q^2$

By Pythagoras, $r^2+p^2 = q^2$,

so $3q^2 = 9x^2 + 9y^2 - 2q^2$,

so $5q^2 = 9(x^2+y^2)$.