SYDNEY GRAMMAR SCHOOL HALF-YEARLY EXAMINATION 2004

FORM V MATHEMATICS & EXTENSION 1

Time allowed: 2 hours

Exam date: 12th May 2004

Instructions

All questions may be attempted.
All questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.

Collection

The writing booklets will be collected in one bundle.
Start each question in a new answer booklet.
If you use a second booklet for a question, place it inside the first. Don't staple.

5A: DS	5B: PKH	5C: DNW	5D: JNC
5E: KWM	5F: BDD	5G: REN	5H: MLS

Checklist

Folded A3 writing booklets required — 8 per boy.
Candidature: 136 boys.

Examiner

BDD

QUESTION ONE Use a separate writing booklet.

(a) Find $f(3)$, where $f(x) = x^2 - 2x + 1$.

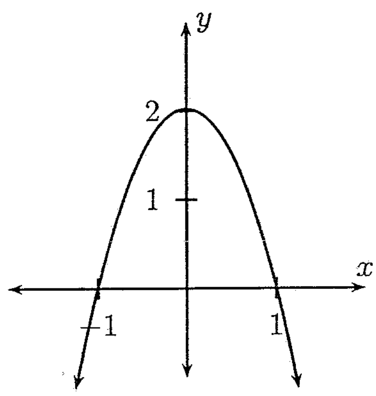
(b) Write $\dfrac{8}{5+\sqrt{3}}$ with a rational denominator in simplest form.

(c) Find a and b, if $a + b\sqrt{3} = 3 + \sqrt{12}$ and a and b are rational.

(d) (i) Sketch the parabola $y = (x-1)(x+4)$, being careful to show x-intercepts and the y-intercept (but the vertex is not required).

(ii) Hence, or otherwise, solve $(x-1)(x+4) \le 0$.

(e)



A parabola with vertex $(0, 2)$ is sketched above.

(i) Explain clearly and briefly why the graph is the graph of a function.

(ii) Write down the range of the function.

(f) A line has an angle of inclination $140°$. What is its gradient? (Give your answer correct to two decimal places.)

(g) Write down (but don't bother simplifying) the equations of the curves resulting when:

(i) $y = x^2$ is shifted 4 units down,

(ii) $y = x^2$ is shifted 1 unit left.

QUESTION TWO Use a separate writing booklet.

(a) Write down the natural domain of $y = \sqrt{x-3}$.

(b) Solve:

(i) $3 - 5x < 13$

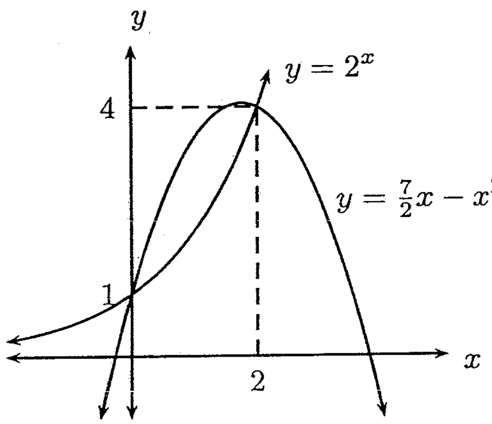
(ii) $|2x + 6| < 4$

(c) Find the inverse function of $f(x) = 4x - 3$.

(d) (i) Sketch $y = 2x - 1$, being careful to mark any intercepts with the axes.

(ii) On the same number plane, sketch $y = |2x - 1|$, using a different colour, or some other method, to distinguish the graphs.

(e)



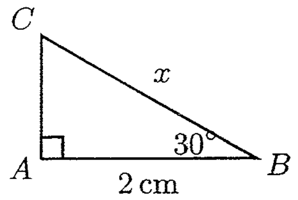
Using the graph above, write down the solution to the inequation $2^x > \frac{7}{2}x - x^2 + 1$.

(f) Solve $\dfrac{4}{x-1} < 1$.

QUESTION THREE Use a separate writing booklet.

(a) Solve $\cos\theta = -\dfrac{\sqrt{3}}{2}$, for $0^\circ \le \theta \le 360^\circ$.

(b)

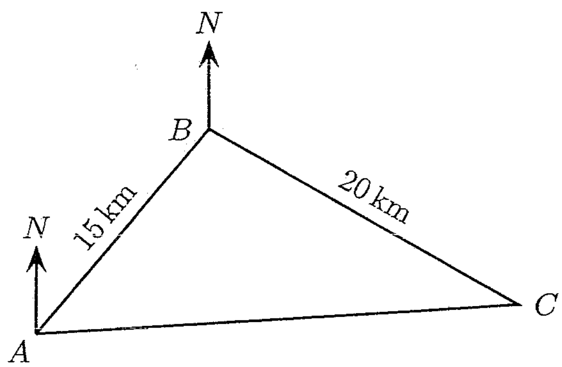


Find the exact length of the side BC, marked x in the diagram above.

(c) Solve $2\cos^2\theta - \cos\theta - 1 = 0$, for $0^\circ \le \theta \le 360^\circ$.

(d) Find the exact value of $\tan\theta$, given that θ is a reflex angle with $\cos\theta = \dfrac{2}{5}$.

(e)



In the diagram above, a ship leaves port A at a bearing of 040°. After travelling 15 km, it arrives at B. The ship then sails at a bearing of 120° for another 20 km, eventually arriving at point C.

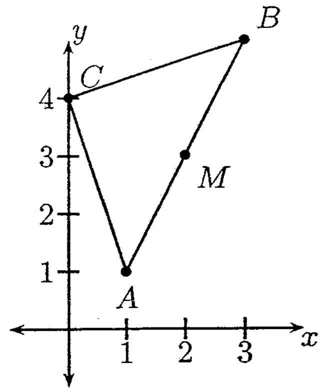
(i) Show, using a diagram or otherwise, that $\angle ABC = 100^\circ$.

(ii) Use the cosine rule to calculate how far the ship is from A when it arrives at C. Give your answer correct to the nearest kilometre.

Exam continues next page ...

QUESTION FOUR Use a separate writing booklet.

(a)



Let $A(1,1)$, $B(3,5)$ and $C(0,4)$ be three vertices of a triangle ABC and $M(2,3)$ be the midpoint of AB, as in the diagram above.

(i) Find the gradient of MC.

(ii) Find the equation of the line MC in general form.

(iii) Show that AB is perpendicular to MC.

(iv) Find the lengths of the intervals AB and MC.

(v) Find the area of $\triangle ABC$.

(b) The lines $2x + 3y - 4 = 0$ and $x + 2y - 6 = 0$ intersect at A. Without finding the coordinates of A, find the equation of the line through A and $(-3, 4)$.

(c) (i) Find the perpendicular distance from the point $(-7, 9)$ to the line $3x - 4y - 18 = 0$.

(ii) How many points of intersection does the line $3x - 4y - 18 = 0$ have with the circle $(x + 7)^2 + (y - 9)^2 = 225$? (A brief reason is required.)

QUESTION FIVE Use a separate writing booklet.

(a) Calculate $\displaystyle\sum_{k=2}^{4} k^2$.

(b) The first few terms of a certain arithmetic sequence are 2, 5, 8, ...

(i) Find the fiftieth term.

(ii) How many terms are less than 500?

(c) Calculate the sum of the first 200 terms of the arithmetic series $1 + 5 + 9 + \cdots$.

(d) Factorise $x^5 - 32$ as the product of two non-trivial factors.

(e) It is known for a certain GP that the fifth term is 9 and the ninth term is 729. Find its first term and common ratio.

Exam continues overleaf ...

QUESTION SIX Use a separate writing booklet.

(a) Prove the trigonometric identity

$$\frac{1}{1+\cos\theta}+\frac{1}{1-\cos\theta}=2\operatorname{cosec}^2\theta\,.$$

(b) The sum of the first n terms of a certain series is given by the formula $S_n = 4n^2+5n$. Find a simplified formula for the nth term T_n of the sequence, for $n \geq 2$. You may use the formula $T_n = S_n - S_{n-1}$.

(c) Let $A(1,5)$ and $B(4,-2)$ be two points in the number plane. Find the coordinates of the point P that divides AB externally in the ratio $2:3$.

(d) A wholesale bakery makes pasties and pies, sold separately in boxes of 20. The Quality Control Manager wishes to model their daily production. The bakery makes more than 6 boxes in total. It requires 3 kg of meat to make a box of pasties and 5 kg of meat to make a box of pies. On any given day, the shop has 30 kg of meat available.

Let the numbers of boxes of pasties produced be x and the numbers of boxes of pies produced be y. The problem of how many pasties and pies to make may be modelled by the equations

$$3x+5y \leq 30$$
$$\text{and}\quad x+y>6\,.$$

(i) Explain why it may also be assumed that $x \geq 0$ and $y \geq 0$.

(ii) Shade the region in the plane represented by the four conditions $3x+5y \leq 30$ and $x+y>6$ and $x \geq 0$ and $y \geq 0$. (Your lines should be drawn neatly and clearly.)

(iii) Assume that the bakery must produce at least one box of pies and at least one box of pasties. Mark clearly ONE point on your number plane that represents a possible solution to the bakery's production for the day, and write down the coordinates of your point.

QUESTION SEVEN Use a separate writing booklet.

(a) Use induction to prove that for all positive integers n,

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2n-1)\times(2n+1)}=\frac{n}{2n+1}\,.$$

(b) (i) Prove that, provided the limiting sum exists,

$$1+\cos\theta+\cos^2\theta+\cdots=(1+\cos\theta)\operatorname{cosec}^2\theta.$$

(ii) For what values of θ does the limiting sum in part (i) not exist?

Exam continues next page ...

(c) Consider the function $f(x) = \dfrac{x^2+1}{x^2-1}$ and the graph of $y = f(x)$.

(i) Show that $f(x)$ is even.

(ii) Find any horizontal and vertical asymptotes.

(iii) Find any x-intercepts and y-intercepts.

(iv) Calculate $f(2)$.

(v) Hence draw the graph, being careful to show the above features.

QUESTION EIGHT Use a separate writing booklet.

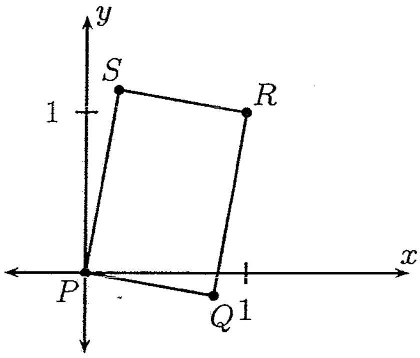
(a) Consider the function $f(x) = |x-3| - |x+3|$.

(i) Is $f(x)$ even, odd, or neither? You must justify your answer.

(ii) Sketch the graph of $y = f(x)$.

(b) The geometric series $1 + \frac{3}{4} + \frac{9}{16} + \cdots$ has limiting sum 4. How many terms must be added before the partial sum differs from 4 by less than 0·0001?

(c)



A certain rectangle $PQRS$ has vertices $P(0,0)$ and $R(1,1)$. The line PS has gradient $m > 1$. The situation is sketched in the diagram above.

(i) Find the lengths of RS and RQ. (HINT: Find the distance from R to PS and from R to PQ.)

(ii) Hence show that the area of $PQRS$ is $\dfrac{m^2-1}{m^2+1}$.

(iii) Let θ be the acute angle between the diagonals of $PQRS$. Find the length PR and show that the area of $PQRS$ is $\sin\theta$. (HINT: You may use the formula $A = \frac{1}{2}ab\sin C$ for the area of a triangle.)

(iv) Use the angle of inclination ϕ of PS and the results above to deduce the identity

$$\cos 2\phi = \frac{1-\tan^2\phi}{1+\tan^2\phi}$$

for $45° < \phi < 90°$. (HINT: First show that $\phi = \frac{\theta}{2} + 45°$.)

END OF EXAMINATION

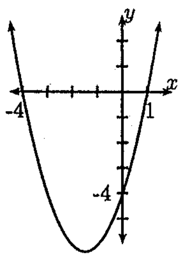
QUESTION ONE

(a) $f(3) = 4$

(b) $\dfrac{8}{5+\sqrt{3}} = \dfrac{8}{5+\sqrt{3}} \times \dfrac{5-\sqrt{3}}{5-\sqrt{3}}$
$= \dfrac{40-8\sqrt{3}}{22}$
$= \dfrac{20-4\sqrt{3}}{11}$

(c) $a = 3,\ b = 2$

(d) (i)



(ii) $-4 \le x \le 1$

(e) (i) There is no vertical line cutting the graph more than once (the vertical line test).

(ii) $y \le 2$

(f) gradient$= \tan 140° \doteqdot -0.84$

(g) (i) $y = x^2 - 4$

(ii) $y = (x+1)^2$

QUESTION TWO

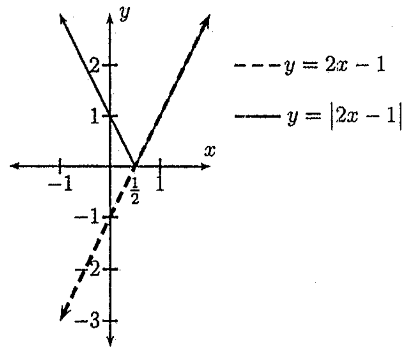
(a) $x \ge 3$

(b) (i) $x > -2$

(ii) Dividing by 2 it follows that $|x+3| < 2$, thus $-5 < x < -1$.

(c)
$$y = 4x - 3$$
$$x = 4y - 3$$
$$y = \tfrac{1}{4}(x+3)$$
$$f^{-1}(x) = \tfrac{1}{4}(x+3)$$

(d)



(e) $x < 0$ or $x > 2$

(f) $\dfrac{4}{x-1} > 1$
$$4(x-1) > (x-1)^2$$
$$0 < (x-1)^2 - 4(x-1)$$
$$0 < (x-1)(x-1-4)$$
$$0 < (x-1)(x-5)$$
From a graph, or otherwise, it follows that $x < 1$ or $x > 5$.

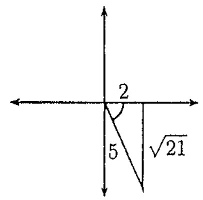
QUESTION THREE

(a) From a ray diagram, θ is quadrant 2 or 3. The related angle is 30°. Thus $\theta = 150°$ or 210°.

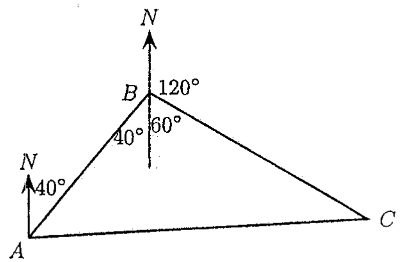
(b) $\cos 30° = \dfrac{2}{x}$
$$x = \frac{2}{\cos 30°}$$
$$x = \frac{4}{\sqrt{3}}$$
$$x = \frac{4\sqrt{3}}{3}$$

(c) $(2\cos\theta + 1)(\cos\theta - 1) = 0$
Hence $\cos\theta = -\frac{1}{2}$ or $\cos\theta = 1$.
The equation $\cos\theta = -\frac{1}{2}$, $0 \le \theta \le 360$ has solutions $\theta = 120°$ or $\theta = 240°$
The equation $\cos\theta = 1$, $0 \le \theta \le 360$ has solutions $\theta = 0°, 360°$.
Thus the solutions of the original equation are $\theta = 0°$, 120°, 240° or 360°.

(d) The angle θ is quadrant 4 (reflex, with $\cos\theta > 0$).
From the diagram, we see that
$$\tan\theta = -\frac{\sqrt{21}}{2}.$$



(e) (i)



From the diagram, $\angle ABC = 40° + 60° = 100°$.

(ii) $AC^2 = 15^2 + 20^2 - 2 \times 15 \times 20 \times \cos 100°$
$\doteqdot 729{\cdot}19$
$AC \doteqdot 27\,\text{km}$ (nearest kilometre)

QUESTION FOUR

(a) (i) gradient $MC = \dfrac{4-3}{0-2} = -\dfrac{1}{2}$.

(ii) $y = -\frac{1}{2}m + 4$, which is in general form $x + 2y - 8 = 0$.

(iii) Gradient of $AB = \dfrac{5-1}{3-1} = 2$, hence gradient $MC\times$ gradient $AB = -1$. Thus the two lines are perpendicular.

(iv)
$$\begin{aligned} AB^2 &= (5-1)^2 + (3-1)^2 \\ &= 20 \\ AB &= \sqrt{20} \\ &= 2\sqrt{5} \\ MC^2 &= (4-3)^2 + (2-0)^2 \\ &= 5 \\ MC &= \sqrt{5} \end{aligned}$$

(v)
$$\begin{aligned} \text{Area } \triangle ABC &= \tfrac{1}{2}\text{base} \times \text{height} \\ &= \tfrac{1}{2}AB \times MC \\ &= \tfrac{1}{2}2\sqrt{5} \times \sqrt{5} \\ &= 5 \end{aligned}$$

(b) The equation through A has the form

$$2x + 3y - 4 + k(x + 2y - 6) = 0$$

for some constant k. Since $(-3, 4)$ is on the line,

$$\begin{aligned} 2(-3) + 3(4) - 4 + k(-3 + 2(4) - 6) &= 0 \\ 2 + k(-1) &= 0 \\ k &= 2 \end{aligned}$$

Thus the equation is

$$\begin{aligned} 2x + 3y - 4 + 2(x + 2y - 6) &= 0 \\ 4x + 7y - 16 &= 0 \end{aligned}$$

(c) (i)
$$\begin{aligned} \text{Distance } &= \frac{|3(-7) - 4(9) - 18|}{\sqrt{3^2 + 4^2}} \\ &= \frac{70}{5} \\ &= 15 \end{aligned}$$

(ii) Since the radius of the given circle is 15, and since the lines is 15 units from the centre of the circle, it must be a tangent. That is, the line intersects the circle exactly once.

QUESTION FIVE

(a) $\displaystyle\sum_{k=2}^{4} k^2 = 2^2 + 3^2 + 4^2 = 29$.

(b) (i) The first term $a = 2$ and the common difference $d = 3$. Thus the fiftieth term is

$$\begin{aligned} T_n &= a + (n-1)d \\ &= 2 + 49 \times 3 \quad (\text{when } n = 50) \\ &= 149 \end{aligned}$$

(ii) We have to solve

$$\begin{aligned} T_n &< 500 \\ 2 + 3(n-1) &< 500 \\ 3n - 1 &< 500 \\ n &< 167 \end{aligned}$$

Hence the first 166 terms are less than 500.

(c) This is the sum of an AP with first term $a = 1$ and common difference $d = 4$. The sum of the first 200 terms is

$$\begin{aligned} S_n &= \tfrac{n}{2}(2a + (n-1)d) \\ &= 100 \times (2 + 199 \times 4) \\ &= 79\,800 \end{aligned}$$

(d)
$$\begin{aligned} x^5 - 32 &= x^5 - 2^5 \\ &= (x-2)(x^4 + 2x^3 + 2^2x^2 + 2^3x + 2^4) \\ &= (x-2)(x^4 + 2x^3 + 4x^2 + 8x + 16) \end{aligned}$$

(e) Given that the nth term of a GP is $T_n = ar^{n-1}$, it means that $ar^4 = 9$ and $ar^8 = 729$. Dividing these equations gives $r^4 = 729 \div 9 = 81$. Hence the common ratio $r = 3$ or $r = -3$. In either case, the first term is $a = 9 \div r^4 = 9 \div 81 = \frac{1}{9}$.

QUESTION SIX

(a) $\text{LHS} = \dfrac{1}{1+\cos\theta} + \dfrac{1}{1-\cos\theta}$

$= \dfrac{1-\cos\theta+1+\cos\theta}{1-\cos^2\theta}$

$= \dfrac{2}{\sin^2\theta}$

$= 2\operatorname{cosec}\theta$

$= \text{RHS}$

(b) By substitution, $S_{n-1} = 4(n-1)^2 + 5(n-1)$. Hence if $n \geq 2$,

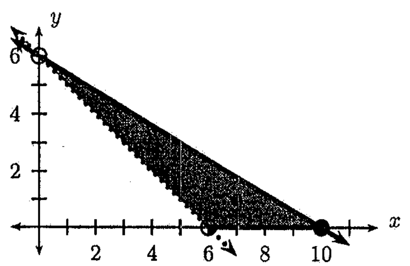
$$\begin{aligned}T_n &= S_n - S_{n-1}\\ &= 4n^2+5n-\left[4(n-1)^2+5(n-1)\right]\\ &= 4n^2+5n-\left[4n^2-8n+4+5n-5\right]\\ &= 4n^2+5n-4n^2+8n-4-5n+5\\ &= 8n+1\end{aligned}$$

(c) This is *external* division, so $AP : PB = 2 : -3$. The point $P(x, y)$ has coordinates;

$$x = \frac{1\times -3 + 2\times 4}{2+-3} = -5 \qquad y = \frac{5\times -3+2\times -2}{2+-3} = 19.$$

(d) (i) In the context, a negative number of boxes makes no sense.

(ii)



(iii) There are a number of answers. One is the point $(7, 1)$, marked on the diagram above.

QUESTION SEVEN

(a) **Step A:** If $n = 1$ then

$\text{LHS} = \dfrac{1}{1\times 3}$

$\text{RHS} = \dfrac{1}{2\times 1+1}$

Hence the results holds for $n = 1$.

Step B: Assume the result holds for $n = k$. That is,

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2k-1)\times(2k+1)} = \frac{k}{2k+1}.$$

We want to prove the result holds for $n = k+1$. That is,

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots+\frac{1}{(2(k+1)-1)\times(2(k+1)+1)} = \frac{k+1}{2(k+1)+1}.$$

$$\begin{aligned}\text{LHS} &= \frac{1}{1\times3}+\cdots+\frac{1}{(2k-1)\times(2k+1)}+\frac{1}{(2k+1)\times(2k+3)}\\ &= \frac{k}{2k+1}+\frac{1}{(2k+1)\times(2k+3)}\\ &= \frac{k(2k+3)+1}{(2k+1)(2k+3)}\\ &= \frac{2k^2+3k+1}{(2k+1)(2k+3)}\\ &= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}\\ &= \frac{(k+1)}{(2k+3)}\\ &= \text{RHS}\end{aligned}$$

Step C: The result follows for all positive integers n by mathematical induction.

(b) (i) This is a geometric series with first term $a = 1$ and common ratio $r = \cos\theta$. The limiting sum is,

$$\begin{aligned}\frac{a}{1-r} &= \frac{1}{1-\cos\theta}\\ &= \frac{1}{(1-\cos\theta)}\times\frac{(1+\cos\theta)}{(1+\cos\theta)}\\ &= \frac{1+\cos\theta}{1-\cos^2\theta}\\ &= \frac{1+\cos\theta}{\sin^2\theta}\\ &= (1+\cos\theta)\operatorname{cosec}^2\theta\end{aligned}$$

(ii) The limiting sum exists (and the formula in part (i) is valid) only if the common ratio r satisfies $|r| < 1$. That is, if $\cos\theta \neq \pm 1$. Thus the limiting sum doesn't exist for $\theta = 0, 180, -180, 360, -360, \ldots$ (i.e. for any multiple of 180°).

(c) (i) $f(-x) = \dfrac{(-x)^2+1}{(-x)^2-1}$

$= \dfrac{x^2+1}{x^2-1}$

$= f(x)$

Hence the function is even.

(ii) The graph has vertical asymptotes where $x^2 - 1 = 0$, i.e. when $x = \pm 1$. The vertical asymptotes are the vertical lines $x = 1$ and $x = -1$.
To find the horizontal asymptotes we divide top and bottom by x^2, thus;

$$\frac{x^2+1}{x^2-1} = \frac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}}$$

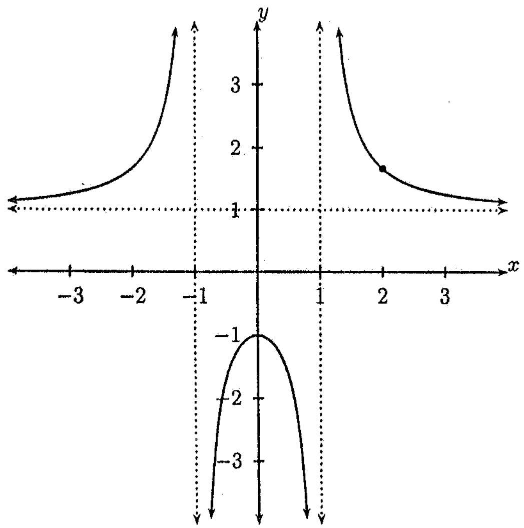
$$\to 1 \quad \text{as } x \to \pm\infty$$

Thus the horizontal asymptote is $y = 1$.

(iii) Put $x = 0$, then $y = -1$. The y-intercept is $(0, -1)$.
Put $y = 0$ then to find the x-intercepts we have to solve $x^2 + 1 = 0$. This has no solutions, so there are no x-intercepts.

(iv) $f(2) = \dfrac{2^2+1}{2^2-1} = \dfrac{5}{3}$.

(v)



QUESTION EIGHT

(a) (i) $f(-x) = |-x-3| - |-x+3|$

$= \big|-(x+3)\big| - \big|-(x-3)\big|$

$= |x+3| - |x-3|$

$= -f(x)$

Hence $f(x)$ is odd.

(ii) For $x \geq 3$, $y = x - 3 - (x+3)$

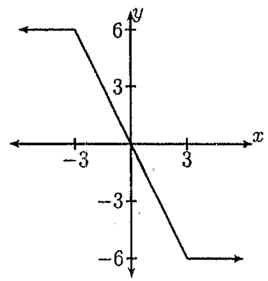
$y = -6$

For $-3 \leq x \leq 3$, $y = -(x-3) - (x+3)$

$y = -2x$

For $x \leq -3$, $y = -(x-3) + x + 3$

$y = 6$



(b) The GP has first term $a = 1$ and common ratio $r = \frac{3}{4}$. The sum of the first n terms is

$$S_n = \frac{a(1-r^n)}{1-r}$$

which differs from the limiting sum $S_\infty = \dfrac{a}{1-r}$ by $\dfrac{ar^n}{1-r} = 4 \times \left(\frac{3}{4}\right)^n$. Thus

$$4 \times \left(\tfrac{3}{4}\right)^n < 0{\cdot}0001$$

$$\left(\tfrac{3}{4}\right)^n < 0{\cdot}000025$$

$$n > \log_{\frac{3}{4}} 0{\cdot}000025$$

$$n > \frac{\log_{10} 0{\cdot}000025}{\log_{10} \frac{3}{4}}$$

$$n > 36.83$$

Hence we must add more than 36 terms (at least 37) before the partial sum differs from 4 by less than 0·0001.

(c) (i) The line PS has gradient m and hence equation $mx - y = 0$. The length RS is the perpendicular distance from $R(1,1)$ to $mx - y = 0$, which is

$$\frac{|m-1|}{\sqrt{m^2+(-1)^2}} = \frac{m-1}{\sqrt{m^2+1}}$$

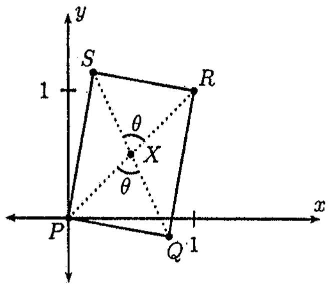
(remember $m > 1$). Now the line PQ has gradient $-\frac{1}{m}$ and hence equation $x+my = 0$. The length RQ is the perpendicular distance from $R(1,1)$ to $x+my = 0$, which is

$$\frac{|m+1|}{\sqrt{m^2+1^2}} = \frac{m+1}{\sqrt{m^2+1}}.$$

(ii) The area of the rectangle $PQRS$ is

$$\begin{aligned} RS \times RQ &= \frac{m-1}{\sqrt{m^2+1}} \times \frac{m+1}{\sqrt{m^2+1}} \\ &= \frac{m^2-1}{m^2+1} \end{aligned}$$

(iii)



By the distance formula, $PR^2 = (1-0)^2 + (1-0)^2$, hence $PR = \sqrt{2}$. Let X be the point of intersection of the diagonals of the square $PQRS$.

Now $PX = XR = XS = XQ = \frac{\sqrt{2}}{2}$. The triangles PXR and RXS each have area

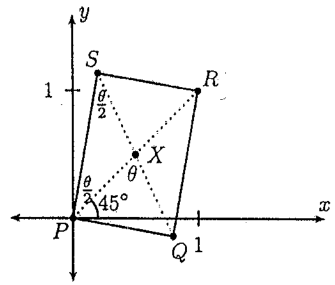
$$\frac{1}{2} \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} \times \sin\theta = \tfrac{1}{4}\sin\theta\,.$$

and the triangles PXS and RXQ each have area

$$\begin{aligned} \frac{1}{2} \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} \times \sin(180-\theta) &= \tfrac{1}{4}\sin(180-\theta) \\ &= \tfrac{1}{4}\sin\theta\,. \end{aligned}$$

The total area of the rectangle is thus $\sin\theta$.

(iv)



The triangle PXS is isosceles and the base angles XSP and XPS add to the exterior angle θ. Hence $\angle XSP = \angle XPS = \frac{\theta}{2}$. Note also that PR is inclined at $45°$ to the x axis. Hence the angle of inclination ϕ of $PS = 45° + \frac{\theta}{2}$. Consider now the identity to be proven;

$$\begin{aligned} \cos 2\phi &= \cos\left(2(\tfrac{\theta}{2} + 45°\right) \\ &= \cos(\theta + 90°) \\ &= \sin(-\theta) \\ &= -\sin\theta \end{aligned}$$

But from part (iii) $\sin\theta$ is the area of $PQRS$, and hence using part (ii);

$$\begin{aligned} \cos 2\phi &= -\frac{m^2-1}{m^2+1} \\ &= \frac{1-m^2}{1+m^2} \\ &= \frac{1-\tan^2\phi}{1+\tan^2\phi} \end{aligned}$$

since the angle of inclination ϕ of PS is related to the gradient by the formula $m = \tan\phi$. (The identity established holds for any angle ϕ, but we have only considered the case $m > 1$ in the diagrams above, so we have only proved the identity when $45° < \phi < 90°$.)

END OF EXAMINATION