Sydney Grammar School
Mathematics Department
Half-Yearly Examinations 2005

FORM V

MATHEMATICS & EXTENSION 1

Examination date

Wednesday 11th May 2005

Time allowed

2 hours

Instructions

All seven questions may be attempted.
All seven questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.

Collection

Write your name, class and master clearly on each booklet.
Hand in the seven questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

5A: WMP	5B: BDD	5C: GJ	5D: MLS
5E: FMW	5F: TCW	5G: JCM/BJC	5H: REP

Checklist

Folded A3 booklets: 7 per boy. A total of 1250 booklets should be sufficient.
Candidature: 137 boys.

Examiner

GJ

QUESTION ONE (15 marks) Use a separate writing booklet. **Marks**

(a) Simplify:

(i) $\sqrt{32}-\sqrt{8}$ $\boxed{1}$

(ii) $(3\sqrt{2}+5)^2$ $\boxed{1}$

(iii) $\dfrac{3\sqrt{27}\times\sqrt{6}}{7\sqrt{18}}$ $\boxed{2}$

(b) Express $0{\cdot}\dot{3}\dot{6}$ as a fraction in simplest form. You must show your working. $\boxed{2}$

(c) Solve the following inequalities:

(i) $3(2-x)\le 5$ $\boxed{1}$

(ii) $|x-3|>9$ $\boxed{2}$

(iii) $x^2-3x<4$ $\boxed{3}$

(d) Given that $\dfrac{\sqrt{3}}{2-\sqrt{3}}=a+b\sqrt{3}$, where a and b are rational, find the values of a and b. $\boxed{3}$

QUESTION TWO (15 marks) Use a separate writing booklet. **Marks**

(a) Find the equation of the line that is perpendicular to $2x-y+1=0$ and passes through the point $(0,-3)$. $\boxed{3}$

(b) Find the angle between the line $x-\sqrt{3}\,y-2\sqrt{3}=0$ and the x-axis. $\boxed{2}$

(c) (i) Show that the point $(2,-1)$ lies on the line $5x-2y-12=0$. $\boxed{1}$

(ii) Hence find the perpendicular distance between the parallel lines $5x-2y-12=0$ and $10x-4y+5=0$. $\boxed{2}$

(d) Without finding the point of intersection, find the equation of the line that passes through the point of intersection of the lines $\boxed{4}$

$$2x-4y+1=0 \qquad \text{and} \qquad x-3y+2=0$$

and is parallel to the line $x+2y+3=0$. Give your answer in general form.

(e) Use the ratio division formula to find the ratio in which the point $P(-8,7)$ divides the interval AB, where A and B are the points $(-3,3)$ and $(7,-5)$ respectively. $\boxed{3}$

Exam continues next page ...

QUESTION THREE (15 marks) Use a separate writing booklet. **Marks**

(a) State the largest possible domain for $y = \dfrac{x}{\sqrt{4-x}}$. $\boxed{2}$

(b) Sketch the following graphs on separate number planes. Show all asymptotes and all intercepts with the x and y axes:

(i) $y = -\sqrt{9-x^2}$ $\boxed{2}$

(ii) $y = \dfrac{2}{x-1}$ $\boxed{2}$

(iii) $y = 3^x - 1$ $\boxed{2}$

(c) A function $f(x)$ is given by $f(x) = \begin{cases} |x+4|\,, & \text{for } x < 0, \\ 4 - x^2, & \text{for } x \geq 0. \end{cases}$

(i) Draw a neat sketch of the function giving all intercepts with the x and y axes. $\boxed{3}$

(ii) Find the values of a for which $f(a) = 2$. $\boxed{2}$

(d) Prove that $f(x) = \dfrac{2^{2x}+1}{2^x}$ is an even function. $\boxed{2}$

QUESTION FOUR (15 marks) Use a separate writing booklet. **Marks**

(a) Simplify fully $\dfrac{3^n + 3^{n+2}}{3^{n+3} + 3^{n+1}}$. $\boxed{2}$

(b) Express $2\log_2 x - \log_2 y = \log_2 z$ as a relation that does not involve logarithms. $\boxed{3}$

(c) Evaluate $\displaystyle\sum_{r=1}^{5} (-1)^{r+1} 2^r$. $\boxed{2}$

(d) An arithmetic series is given by $1{\cdot}25 + 1{\cdot}21 + 1{\cdot}17 + \cdots$.

(i) Find the 121st term of the series. $\boxed{3}$

(ii) Find the sum of the first 121 terms. $\boxed{2}$

(e) Consider the geometric series $\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots$.

(i) Find the sum S_9 of the first 9 terms of the series. $\boxed{2}$

(ii) If $S_9 = 2 - 2^k 3^\ell$, where k and ℓ are integers, find the values of k and ℓ. $\boxed{1}$

Exam continues overleaf ...

QUESTION FIVE (15 marks) Use a separate writing booklet. **Marks**

(a) Find the exact value of the product $\sin 60^\circ \sin 315^\circ$, rationalising the denominator. $\boxed{3}$

(b) If $\sin\theta = -\frac{3}{4}$ and $\tan\theta < 0$, find the exact value of $\cos\theta$. $\boxed{3}$

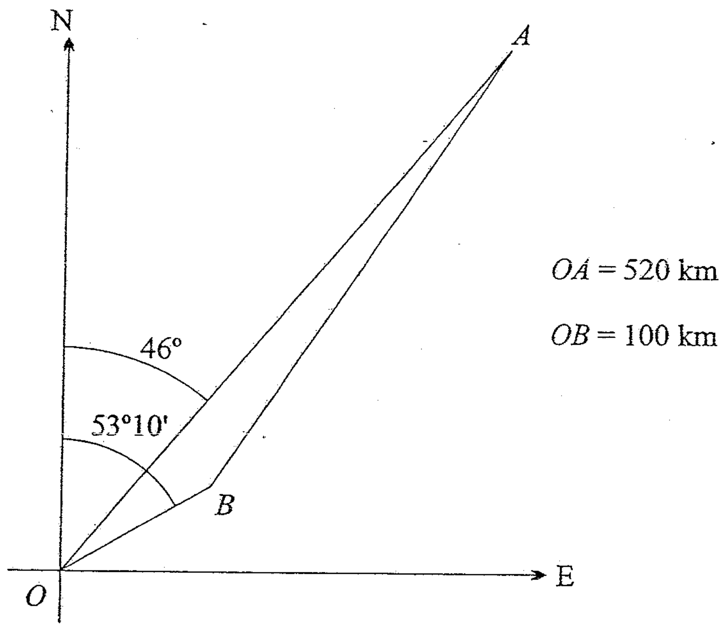
(c) Solve the equation $\boxed{3}$

$$\sin^2\theta + 3\sin\theta - 1 = 0, \quad \text{for } 0^\circ \le \theta \le 360^\circ,$$

giving your answers correct to the nearest minute.

(d) Simplify fully the expression $\sqrt{(1-\sin^2 x)(1+\tan^2 x)}$, where x is acute. $\boxed{2}$

(e)



A ship normally sails directly from a port at O to a port at A, which is 520 km from O on a bearing of N46°E from O.

To avoid a storm front, the ship has had to sail the first 100 km of its journey on a bearing of N53°10′E to port B where it takes refuge.

(i) How far, correct to the nearest kilometre, is B from A? $\boxed{2}$

(ii) On what bearing must the ship now sail in order to reach A? Give your answer correct to the nearest minute. $\boxed{2}$

QUESTION SIX (15 marks) Use a separate writing booklet. **Marks**

(a) Solve the inequality **4**

$$\frac{x+1}{x-1} \le x+1.$$

(b) Prove the identity **3**

$$\operatorname{cosec}\theta(1+\cot\theta)+\sec\theta(1+\tan\theta)=\frac{\cos\theta+\sin\theta}{\sin^2\theta\cos^2\theta}.$$

(c) The following points are plotted on a number plane:

$P_1(2,1),\ P_2(2\tfrac{1}{2},\tfrac{1}{2}),\ P_3(3,\tfrac{1}{4}),\ \ldots$

Note that the x-values form an AP and the y-values form a GP.

(i) (α) Prove that the nth point P_n is $\left(\dfrac{3+n}{2},\ 2^{1-n}\right)$. **3**

(β) Show that for all positive integers n, the point P_n lies on the graph of the function $y=2^{4-2x}$. **2**

(ii) Bill wants to insert a point between P_1 and P_2 so that $x=2\frac{1}{4}$, which is the arithmetic mean of the x-values of P_1 and P_2. Find the corresponding y-value and show that it is the positive geometric mean of the y-values of P_1 and P_2. **3**

NOTE: Given two positive numbers a and b, their arithmetic mean is $\dfrac{a+b}{2}$ and their positive geometric mean is $\sqrt{ab}$.

QUESTION SEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Shade the region on the number plane satisfying both inequalities **3**

$$y > 4 - x^2 \qquad \text{and} \qquad y \le x^2 - 4.$$

(b) A function is defined by

$$y = \frac{(x-2)(x+2)^2}{x^4}.$$

(i) Write down the domain of the function. **1**

(ii) Find any intercepts with the x-axis. **1**

(iii) Write down the equations of any horizontal or vertical asymptotes. **2**

(iv) Copy and complete the following table of values. Give your answers correct to two decimal places. **2**

x	-3	-2	-1	1	2	3
y						

(v) Draw a neat graph of the function showing all intercepts. **2**

(c) A GP has a second term $T_2 = 1$ and limiting sum $S_\infty = \dfrac{1}{k}$. Find all possible values of k. **4**

END OF EXAMINATION

FORM V 3U HALF YEARLY 2005

QUESTION ONE

(a) (i) $\sqrt{32} - \sqrt{8} = 4\sqrt{2} - 2\sqrt{2}$

$= 2\sqrt{2}$ ✓

(ii) $(3\sqrt{2} + 5)^2 = 18 + 30\sqrt{2} + 25$

$= 43 + 30\sqrt{2}$ ✓

(iii) $\dfrac{3\sqrt{27} \times \sqrt{6}}{7\sqrt{18}} = \dfrac{3\sqrt{27}}{7\sqrt{3}}$

$= \dfrac{9}{7}$ ✓✓

(b) $0.\dot{3}\dot{6} = 0.36 + 0.0036 + 0.000036 + \ldots$

$= \dfrac{0.36}{1 - 0.01}$

$= \dfrac{36}{99}$

$= \dfrac{4}{11}$ ✓✓ (any suitable method)

(c) (i) $3(2-x) \leq 5$

$6 - 3x \leq 5$

$3x \geq 1$

$x \geq \frac{1}{3}$ ✓

(ii) $|x-3| > 9$

$x - 3 < -9$ or $x - 3 > 9$

$x < -6$ or $x > 12$ ✓✓ (any suitable method)

(iii) $x^2 - 3x < 4$

$x^2 - 3x - 4 < 0$ ✓

$(x-4)(x+1) < 0$ ✓

$-1 < x < 4$ ✓ (any suitable method)

(d) $\frac{\sqrt{3}}{2-\sqrt{3}} = a+b\sqrt{3}$

$\frac{\sqrt{3}}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}} = a+b\sqrt{3}$ ✓

$3+2\sqrt{3} = a+b\sqrt{3}$

So $a=3$ and $b=2$ ✓✓

QUESTION TWO

(a) $2x-y+1=0$

$y = 2x+1$

Gradient $=2$ ✓

Gradient of perpendicular

$= -\frac{1}{2}$ ✓

So $y = -\frac{1}{2}x - 3$ ✓

(b) $x-\sqrt{3}y-2\sqrt{3}=0$

so $y = \frac{1}{\sqrt{3}}x - 2$

$m = \frac{1}{\sqrt{3}}$

now $\tan\theta = \frac{1}{\sqrt{3}}$ ✓

so $\theta = 30°$ ✓

(c) (i) LHS $= 5x-2y-12$

$= 10+2-12$

$= 0$ ✓

$=$ RHS

So the point lies on the line $5x-2y-12=0$

c(ii) Distance $= \dfrac{|2\times 10 - 4\times -1 + 5|}{\sqrt{100+16}}$ ✓

$= \dfrac{29}{\sqrt{116}}$

$= \dfrac{29}{2\sqrt{29}}$

$= \dfrac{\sqrt{29}}{2}$ units ✓ (accept either)

(d) $2x-4y+1+k(x-3y+2)=0$, k a constant ✓

$(2+k)x-(4+3k)y+(1+2k)=0$

Gradient $= \dfrac{2+k}{4+3k}$

$x+2y+3=0$

Gradient $= -\dfrac{1}{2}$

Now $\dfrac{2+k}{4+3k} = -\dfrac{1}{2}$ ✓

$4+2k = -4-3k$

$5k = -8$

$k = -\dfrac{8}{5}$ ✓

So $2x-4y+1-\dfrac{8}{5}(x-3y+2)=0$

$10x-20y+5-8x+24y-16=0$

$2x+4y-11=0$ ✓

(c) Let the ratio be $k:l$.

$$x = \frac{kx_2 + lx_1}{k+l}$$

$$-8 = \frac{7k - 3l}{k+l} \quad ✓$$

$$-8k - 8l = 7k - 3l$$
$$-5l = 15k \quad ✓$$
$$\frac{k}{l} = -\frac{1}{3}$$

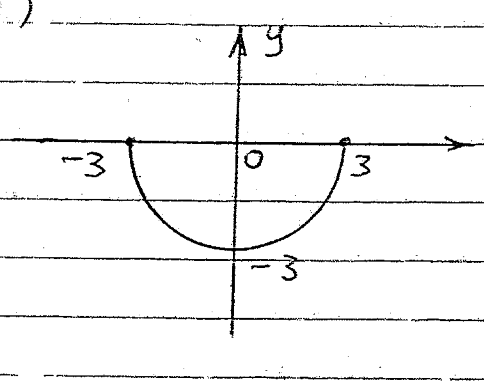
So $k : l = -1 : 3$ ✓

QUESTION THREE

(a) $4 - x > 0$ ✓

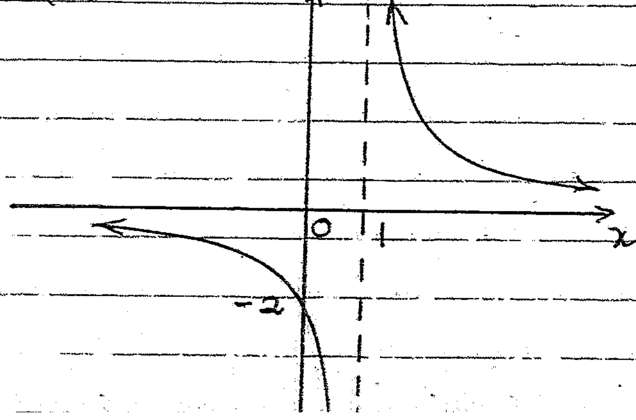
So $x < 4$ ✓

(b) (i)



✓✓

(ii)



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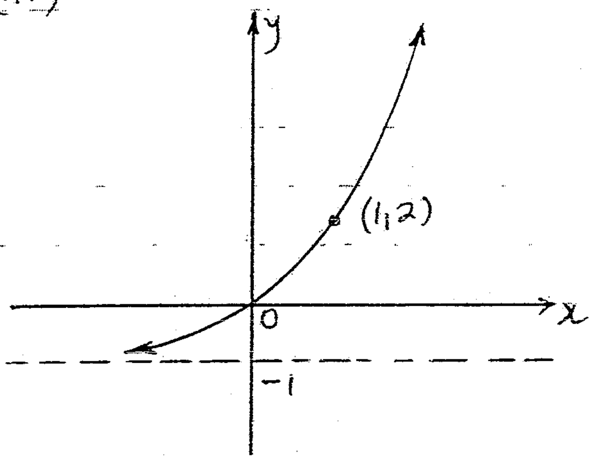
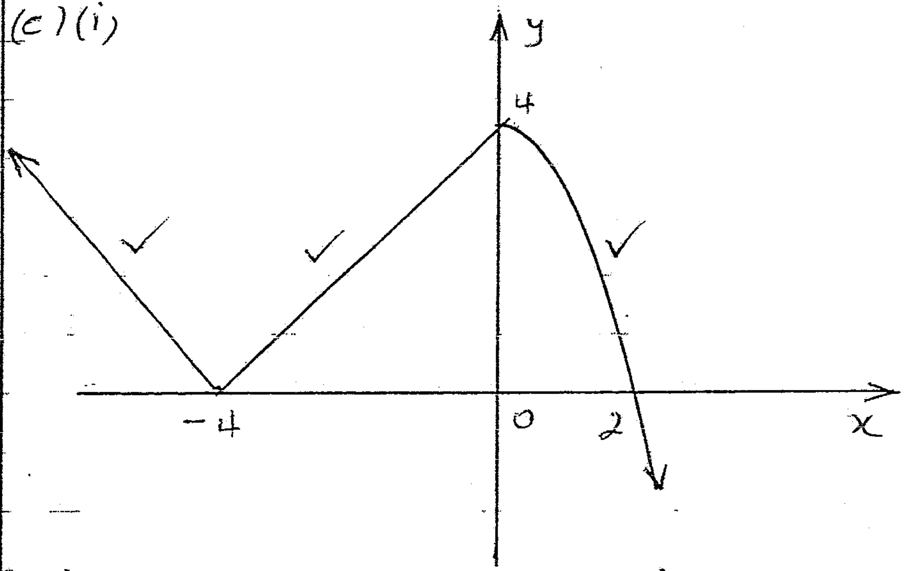
(asymptotes and intercept must be clearly shown)

(iii)



✓✓
(asymptote and intercept must be clearly shown).

(c)(i)



(ii) For $a < 0$
$|a+4| = 2$
So $a = -2$ or -6 ✓

(iii) For $a \geqslant 0$
$4 - a^2 = 2$
$a^2 = 2$
$a = \sqrt{2}$ ✓

(d) $f(x) = \dfrac{2^{2x}+1}{2^x}$ ✓

$f(-x) = \dfrac{2^{-2x}+1}{2^{-x}} \times \dfrac{2^{2x}}{2^{2x}}$

$= \dfrac{1+2^{2x}}{2^x}$ ✓

$= f(x)$

So f(x) is ...

QUESTION FOUR

(a) $\dfrac{3^n+3^{n+2}}{3^{n+3}+3^{n+1}} = \dfrac{3^n(1+3^2)}{3^{n+1}(3^2+1)}$ ✓

$= \dfrac{1}{3}$ ✓

(b) $2\log_2 x - \log_2 y = \log_2 z$

$\log_2 x^2 - \log_2 y = \log_2 z$ ✓

$\log_2\left(\dfrac{x^2}{y}\right) = \log_2 z$ ✓

So $\left.\begin{aligned} \dfrac{x^2}{y} &= z \\ x^2 &= yz \end{aligned}\right\}$ ✓

(c) $\sum_{r=1}^{5} (-1)^{r+1} 2^r = 2 - 4 + 8 - 16 + 32$ ✓

$= 22$ ✓

(d)(i) $a = 1{\cdot}25$, $d = -0{\cdot}04$ ✓✓

$T_{121} = 1{\cdot}25 + 120 \times -0{\cdot}04$

$= -3{\cdot}55$ ✓

(ii) $S_{121} = \dfrac{121}{2}(1{\cdot}25 - 3{\cdot}55)$ ✓

$= -139{\cdot}15$ ✓

(e)(i) $a = \dfrac{2}{3}$, $r = \dfrac{2}{3}$

$S_9 = \dfrac{\frac{2}{3}\left(1-\left(\frac{2}{3}\right)^9\right)}{1-\frac{2}{3}}$ ✓

$= 2\left(1-\left(\frac{2}{3}\right)^9\right)$ ✓

(ii) $S_9 = 2(1 - 2^9 \times 3^{-9})$

$= 2 - 2^{10} \times 3^{-9}$

So $k = 10$ and $l = -9$ ✓

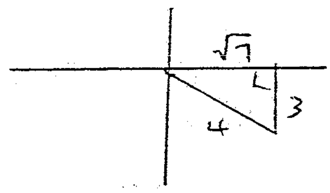
QUESTION FIVE

(a) $\sin 60^\circ \times \sin 315^\circ = \frac{\sqrt{3}}{2} \times -\frac{1}{\sqrt{2}}$ ✓✓

$= -\frac{\sqrt{3}}{2\sqrt{2}}$

$= -\frac{\sqrt{6}}{4}$ ✓

(b) $\cos\theta = \frac{\sqrt{7}}{4}$



✓ quadrant
✓ $\sqrt{7}$
✓ $\frac{\sqrt{7}}{4}$

(c) $\sin^2\theta + 3\sin\theta - 1 = 0$

$\sin\theta = \frac{-3+\sqrt{13}}{2}$ or $\frac{-3-\sqrt{13}}{2}$ ✓

Now $-1 \le \sin\theta \le 1$

So $\sin\theta = \frac{-3+\sqrt{13}}{2}$

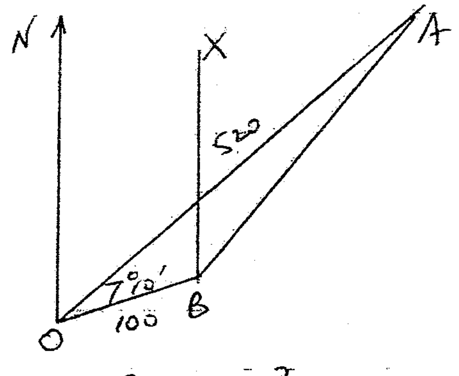
$\theta \doteqdot 17^\circ 37'$ or $162^\circ 23'$ ✓✓

(d) $\sqrt{(1-\sin^2 x)(1+\tan^2 x)} = \sqrt{\cos^2 x \times \sec^2 x}$ ✓

$= \sqrt{1}$

$= 1$ ✓

(e)



(i) $AB^2 = 100^2 + 520^2 - 2 \times 100 \times 520 \times \cos 7^\circ 10'$ ✓

$AB^2 = 177212.50\ldots$

$AB \doteqdot 421$ km ✓

(ii) $\angle OBX = 126°50'$ (cointerior angles, $ON \parallel OX$)

Now $\dfrac{\sin \angle OBA}{520} = \dfrac{\sin 7°10'}{AB}$

So $\angle OBA = 171°8'$ ✓

Bearing angle $ABX = 171°8' - 126°50'$

$= 44°18'$

The bearing is $N 44°18' E$ ✓

If the cosine rule is used to find $\angle OBA$ and AB is rounded the bearing is $N44°8'E$.

QUESTION SIX

(a) $\dfrac{x+1}{x-1} \le x+1$

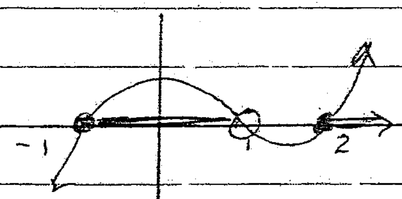
$\dfrac{x+1}{x-1} \times (x-1)^2 \le (x+1)(x-1)^2, \quad x \ne 1$ ✓

$(x+1)(x-1) - (x+1)(x-1)^2 \le 0$

$(x+1)(x-1)(1-x+1) \le 0$

$(x+1)(x-1)(2-x) \le 0$ ✓

So $-1 \le x < 1$ or $x \ge 2$. ✓



METHOD ✓

(b) $LHS = \operatorname{cosec}\theta(1+\cot\theta) + \sec\theta(1+\tan\theta)$

$= \dfrac{1}{\sin\theta}\left(1 + \dfrac{\cos\theta}{\sin\theta}\right) + \dfrac{1}{\cos\theta}\left(1 + \dfrac{\sin\theta}{\cos\theta}\right)$ ✓

$= \dfrac{1}{\sin^2\theta}(\sin\theta + \cos\theta) + \dfrac{1}{\cos^2\theta}(\cos\theta + \sin\theta)$

$= (\cos\theta + \sin\theta)\left(\dfrac{1}{\sin^2\theta} + \dfrac{1}{\cos^2\theta}\right)$

$= (\cos\theta + \sin\theta)\left(\dfrac{\cos^2\theta + \sin^2\theta}{\sin^2\theta\cos^2\theta}\right)$

$= \dfrac{\sin\theta + \cos\theta}{\sin^2\theta\cos^2\theta}$ ✓✓

(c)(i)(α) The x-values form an AP with $a=2$, $d=\frac{1}{2}$

$T_n = a+(n-1)d$
$= 2+\frac{1}{2}(n-1)$ ✓ (for setup)
$= \frac{3+n}{2}$ ✓

The y-values form an AP with $a=1$ and $r=\frac{1}{2}$

$T_n = ar^{n-1}$
$= (\frac{1}{2})^{n-1}$
$= 2^{1-n}$ ✓

So P_n is $\left(\frac{3+n}{2}, 2^{1-n}\right)$

(β) LHS $= y$
$= 2^{1-n}$
RHS $= 2^{4-2x}$
$= 2^{4-2(\frac{3+n}{2})}$
$= 2^{1-n}$

So LHS = RHS ✓✓

So the points lie on the graph of $y=2^{4-2x}$

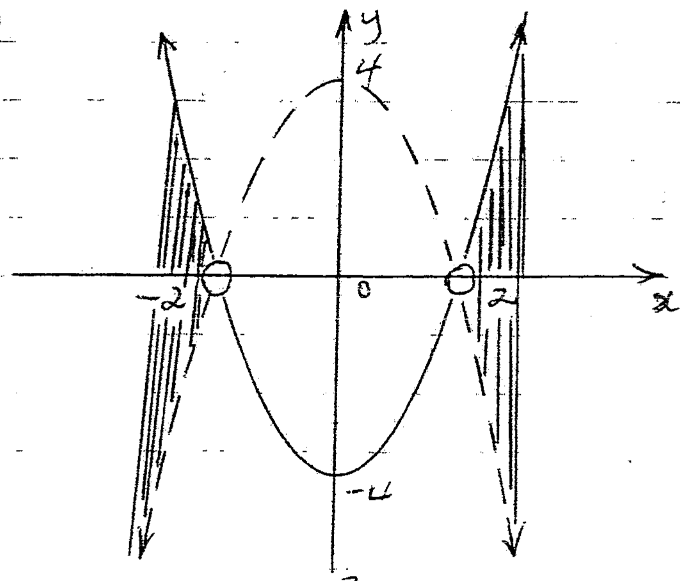
(ii) If $x=2\frac{1}{4}$, $y=2^{4-2\times2\frac{1}{4}}$
$= 2^{-\frac{1}{2}}$
$= \frac{1}{\sqrt{2}}$ ✓

G.M. of 1 and $\frac{1}{2}$ $= \sqrt{1\times\frac{1}{2}}$
$= \frac{1}{\sqrt{2}}$ ✓

So the y-value is a geometric mean ✓

QUESTION SEVEN.

(a)



✓ dotted curve and open circles
✓✓ shading

(b) $y = \frac{(x-2)(x+2)^2}{x^4}$

(i) Domain: $x \neq 0$ ✓

(ii) when $y = 0$, $x = 2$ or -2 ✓

(iii) The vertical asymptote is $x = 0$ ✓

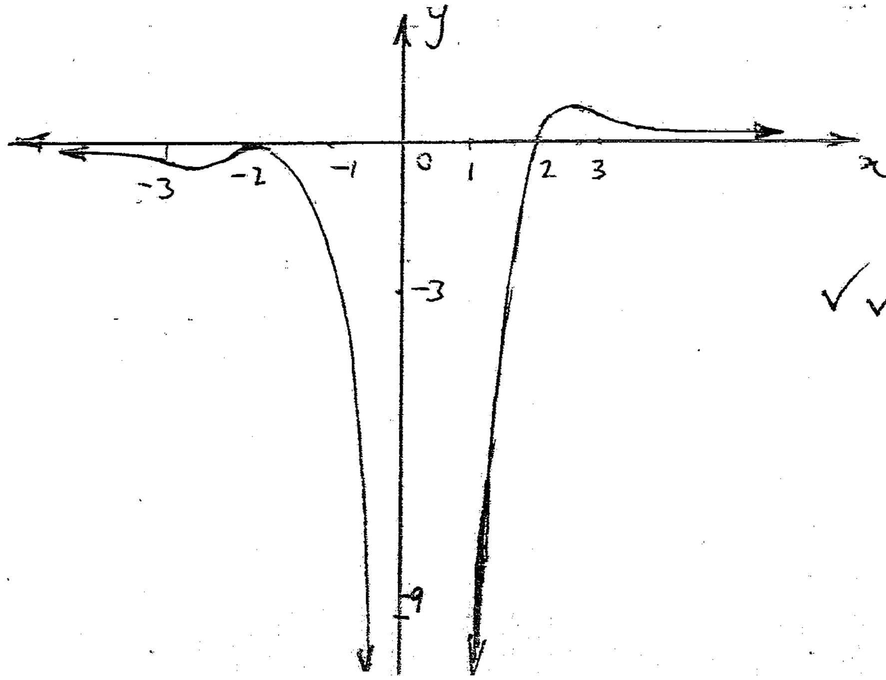
The horizontal asymptote is $y = 0$ ✓

(iv)

x	-3	-2	-1	1	2	3
$f(x)$	-0.06	0	-3	-9	0	0.31

✓✓

(v)



✓✓

(b) $T_2 = 1$

So $ar = 1$ —(1)

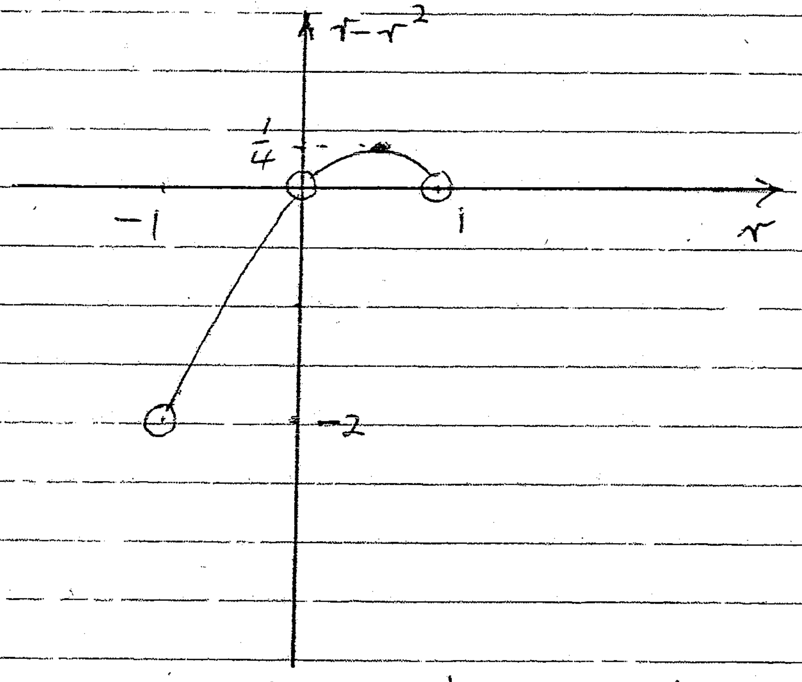
$S_\infty = \frac{1}{k}$

So $\frac{a}{1-r} = \frac{1}{k}$ —(2)

Substitute (1) in (2)

$$\frac{1}{r(1-r)} = \frac{1}{k}$$

$r(1-r) = k$, where $-1 < r < 1$ and $r \neq 0$.



So $-2 < k \leq \frac{1}{4}$ and $k \neq 0$