SYDNEY GIRLS HIGH SCHOOL

YEAR 11 Extention 1 Mathematics

Assessment Task one

November 2004

Time allowed: 75 minutes

Topics: Integration, Locus and sequences and series

Instructions:

- There are three questions. Questions **are** of equal value.
- Attempt all questions.
- Show all necessary working. Marks may be deducted for badly arranged work.
- Start each question on a new page. Write on one side of the paper only.

Name: ....

QUESTION ONE (25 marks)

a) Integrate the following

i) $\int (5+3x^2)\ dx$

ii) $\int (x-4)(x+3)\ dx$

iii) $\int \frac{dx}{(5x+8)^3}$

iv) $\int \frac{1+x^2}{\sqrt{x}}\ dx$

b) For the parabola $(y+3)^2 = 8-8x$ find:

i) the focal length

ii) the coordinates of the vertex

iii) the coordinates of the focus

iv) the equation of the directrix

c) Evaluate $\int_0^{\frac{4}{3}} (1-3x)^3 dx$

d) A point $P(x,y)$ moves so that its distance from the $y-axis$ is equal to its distance from (4,2). Find the equation of the locus of P.

e) Find the values of r for which the geometric series $1+(r+2)+(r+2)^2+$...............has a limiting sum.

f) Find the sum of $\frac{2}{9}+2+18+$...............$+118098$

g) Find the value of k

$$\int_1^k \frac{4}{x^2}\ dx = 3$$

QUESTION TWO (25 marks)

a) A series is defined by $S_n = \sum_{r=1}^{n} 2^{4-r}$

i) Show that the series is geometric

ii) Find the difference between the limiting sum and the sum to six terms

b) Given $f(x) = \dfrac{4}{2^x + 2^{-x}}$ use the Simpson's rule with 5 ordinates to find the approximate area bounded by the curve and the $x-axis$ from $x = 2$ to $x = -2$.

c) Find the coordinates of the vertex and the focus of the parabola $y = x^2 - 6x + 10$

d) Calculate the following definite integral

$$\int_0^1 x^{\frac{1}{2}}(1-x)\, dx$$

e) Find the area enclosed by the curves $y = 6x - x^2$ and $y = x^4 - 6x^3$.

f) A parabola has vertex (-3,-1) and focus (-3,-2); determine its equation

g) The limiting sum of a geometric series is 3, and the sum of its first two terms is $2\frac{2}{3}$.

Show that there are two such series and find their common ratio.

h) Find the coordinates of the centre and the radius of $4x^2 + 4y^2 + 20x = 24y - 25$

QUESTION THREE (25 marks)

a) i) How many terms are there in the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+.............+\frac{1}{2^n}$

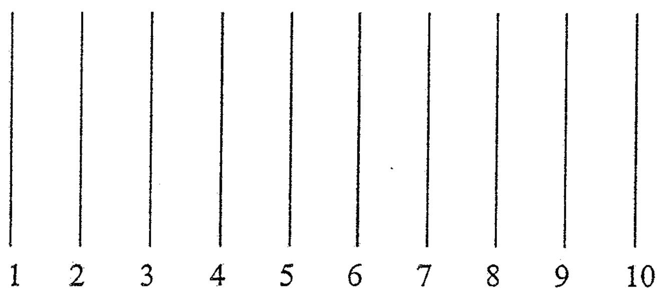
ii) By using the formula for the sum of a geometric series prove

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+.............+\frac{1}{2^n}=2-\frac{1}{2^n}$$

b) Prove by induction that

$$\frac{1}{1\times5}+\frac{1}{5\times9}+\frac{1}{9\times13}+........+\frac{1}{(4n-3)(4n+1)}=\frac{n}{4n+1}$$

c) As part of her training, Mona runs "line sprints". Ten parallel lines 4m apart are drawn on the athletic field. Mona stands at line 1, runs to line 2, runs back to line 1, runs to line 3, back to line 1, runs to line 4, back to line 1 and so on.



i) How far does she run altogether?
ii) How far left to run after she reaches line 7?

d) Consider the points $A(1,0)$ $B(5,0)$ and $P(x,y)$. Given that is PA perpendicular to PB, find the locus of P and describe it geometrically.

e) Toni borrows $240 000 in order to buy a studio. Interest of 12% p.a. on the loan is calculated monthly on the balance still owing. The equal repayments, of $M, are made each monthly and the loan is repaid over 20 years.

i) Show that the amount A_2 still owing after making two repayments of $M each is given by $A_2 = 240000(1.01)^2 - M(1+1.01)$

ii) Find the value of each monthly repayment

iii) How much will she still owe after 5 years of repayments?

f) Find the volume of the solid generated when the area under the curve $y=(x-3)^3$ and the line $x=4$ is rotated

i) about the x-axis

ii) about the y-axis

THE END

Ext 1 Nov 2004

a) i) $5x + \frac{3x^3}{3} + c$

$5x + x^3 + c$ ①

ii) $\int x^2 + 3x - 4x - 12 \, dx$

$\int x^2 - x - 12 \, dx$

$\frac{x^3}{3} - \frac{x^2}{2} - 12x + c$ ②

iii) $\int (5x+8)^{-3} \, dx$

$= \frac{(5x+8)^{-2}}{-2 \times 5} + c$

$= \frac{-1}{10(5x+8)^2} + c$ ②

iv) $\int (1+x^2)\, x^{-\frac{1}{2}} \, dx$

$= \int x^{-\frac{1}{2}} + x^{\frac{3}{2}} \, dx$

$= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c$ ②

$= 2\sqrt{x} + \frac{2\sqrt{x^5}}{5} + c$

$= 2\sqrt{x} + \frac{2x^2\sqrt{x}}{5}$

$= 2\sqrt{x}\left(1 + \frac{x^2}{5}\right) + C$

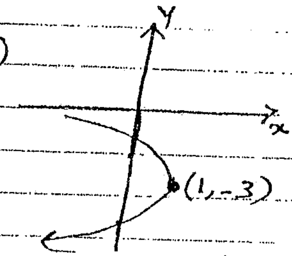
b) i) $(y+3)^2 = -8(x-1)$

$4a = 8 \quad a = 2$ ①

ii) $V(1, -3)$ ①

iii) $F(-1, -3)$ ①

iv) $x = 3$ ①



c) $\int_0^{\frac{4}{3}} (1-3x)^3 \, dx$

$\left[\frac{(1-3x)^4}{-3 \times 4}\right]_0^{\frac{4}{3}}$

$= -\frac{1}{12}\left(\left(1 - 3 \times \frac{4}{3}\right)^4 - 1\right)$

$= -\frac{1}{12}\left((-3)^4 - 1\right)$ ③

$= -\frac{1}{12} \times 80$

$= -6\frac{2}{3}$

d) $\sqrt{(x-0)^2 + (y-y)^2} = \sqrt{(x-4)^2 + (y-2)^2}$

$x^2 = x^2 - 8x + 16 + y^2 - 4y + 4$

$(y-2)^2 = 8x - 16 \qquad (y-2)^2 = 8(x-2)$ ③

e) $|r+2|<1$

$r+2<1$

or ②

$r+2>-1$

$r<-1$ or $r>-3$

f) $a=\frac{2}{9}$ $r=9$

$T_n=\frac{2}{9}\times 9^{n-1}$

$118098=\frac{2}{9}\times 9^{n-1}$ ③

$531441=9^{n-1}$

$9^6=9^{n-1}$

$n-1=6$

$n=7$ $S_7=\frac{\frac{2}{9}(9^7-1)}{9-1}$

$=132860\frac{2}{9}$

g) $\int_1^k \frac{4}{x^2}\,dx=3$

$\int_1^k 4x^{-2}dx=\left[\frac{4x^{-1}}{-1}\right]_1^k=3$

$\left[\frac{-4}{x}\right]_1^k=3$ ③

$\frac{-4}{k}+\frac{4}{1}=3$

$\frac{-4}{k}=-1$ $\therefore k=4$

2) a) i)

$T_1=2^3$

$=8$

$T_2=2^2$

$=4$

$T_3=2$ ②

$\frac{T_1}{T_2}=\frac{8}{4}=2$

$\frac{T_2}{T}=\frac{4}{8}=\frac{1}{2}$

$\therefore r=\frac{1}{2}$ $\therefore$ G.P

ii) $S_6=\frac{8(1-(\frac{1}{2})^6)}{\frac{1}{2}}$

$=\frac{8(1-\frac{1}{64})}{\frac{1}{2}}$

$=16(\frac{63}{64})$

$=15\frac{3}{4}$ ③

$S_\infty=\frac{8}{1-\frac{1}{2}}$

$=16$

$S_\infty-S_6=16-15\frac{3}{4}$

$=\frac{1}{4}$

b)

x	$f(x)$	w	$wf(x)$
-2	$\frac{16}{17}$	1	$\frac{16}{17}$
-1	$1\frac{3}{5}$	4	$6\frac{2}{5}$
0	2	2	4
1	$1\frac{3}{5}$	4	$6\frac{2}{5}$
2	$\frac{16}{17}$	1	$\frac{16}{17}$

$h=1$ ③

$A\doteq\frac{1}{3}\sum w f(x)$

$=6\frac{58}{255}$

$\doteq 6.23$

c) $y-10=x^2-6x$

$y-10+9=x^2-6x+9$

$(x-3)^2=y-1$

$V(3,1)$ ③

$4a=1$

$a=\frac{1}{4}$

$F(3,1\frac{1}{4})$

d) $\int_0^1 x^{\frac{1}{2}}-x^{\frac{3}{2}}\,dx$

$\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1$ ②

$=\frac{2}{3}x^{\frac{3}{2}}-\frac{2}{5}x^{\frac{5}{2}}\Big]_0^1=\frac{4}{15}$

e) $6x-x^2=x^4-6x^3$

$x^4-6x^3+x^2-6x=0$

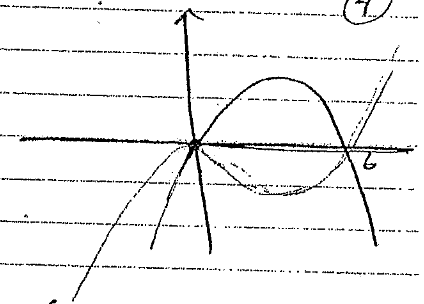
$x^2(x^2-6x)+(x^2-6x)=0$

$(x^2+1)(x^2-6x)=0$

$x^2+1\neq 0$

$x(x-6)=0$

$x=0$ or 6 ④



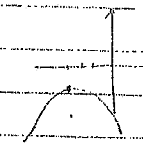
$\left|\int_0^6 x^4-6x^3\,dx-\int_0^6 6x-x^2\right|$

$\left|\frac{x^5}{5}-\frac{6x^4}{4}-\frac{6x^2}{2}+\frac{x^3}{3}\right]_0^6$

$\left|\frac{7776}{5}-\frac{7776}{4}-\frac{216}{2}+\frac{216}{3}\right|$

$A=424\frac{4}{5}\ u^2$

f)



$(x+3)^2=-4(y+1)$

①

g) $S_\infty = 3$

$a + ar = \frac{8}{3}$

$\frac{a}{1-r} = 3$

$a = 3(1-r)$

$a(1+r) = \frac{8}{3}$

$3(1-r)(1+r) = \frac{8}{3}$

$(1-r)(1+r) = \frac{8}{9}$

$1-r^2 = \frac{8}{9}$

$r^2 = \frac{1}{9}$

$r = \pm\frac{1}{3}$ (4)

$r = \frac{1}{3}$

$a = (1-\frac{1}{3})3$

$a = 2$

$r = \frac{-1}{3}$

$a = (1+\frac{1}{3}) \times 3$

$= 4$

$2, \frac{2}{3}, \frac{2}{9}, \ldots$

$4, -1\frac{1}{3}, \frac{4}{9}, \ldots$

h) $4x^2 + 4y^2 + 20x - 24y = -25$

$x^2 + 5x + y^2 - 6y = \frac{-25}{4}$

$(x+\frac{5}{2})^2 + (y-3)^2 = \frac{-25}{4} + \frac{25}{4} + 9$

$C(-\frac{5}{2}, 3)$ $r = 3$

(2)

3 a) i) n+1 terms

(1)

ii) LHS $S_n = \frac{1(1-(\frac{1}{2})^{n+1})}{\frac{1}{2}}$

$= 2(1 - \frac{1}{2^{n+1}})$

$= 2 - \frac{2}{2^{n+1}}$

$= 2 - 2^{-n}$

$= 2 - \frac{1}{2^n}$ (2)

= RHS

b) $n = 1$

LHS $= \frac{1}{5}$

RHS $= \frac{1}{4+1}$

$= \frac{1}{5}$

True for n=1

$n = k$

$\frac{1}{1\times5} + \frac{1}{5\times9} + \frac{1}{9\times13} + \ldots + \frac{1}{(4k-3)(4k+1)} = \frac{k}{4k+1}$

Prove for $n = k+1$

LHS $= \frac{1}{1\times5} + \frac{1}{5\times9} + \ldots + \frac{1}{(4k-3)(4k+1)} + \frac{1}{[4(k+1)-3](4(k+1)+1)}$

RHS $= \frac{k+1}{4(k+1)+1} = \frac{k+1}{4k+5}$

$= \frac{k}{4k+1} + \frac{1}{(4k+1)(4k+5)}$

$= \frac{(4k+5)k+1}{(4k+1)(4k+5)}$

$= \frac{4k^2+5k+1}{(4k+1)(4k+5)}$

$= \frac{4k^2+4k+1k+1}{(4k+1)(4k+5)}$

$= \frac{4k(k+1)+(k+1)}{(4k+1)(4k+5)}$

$= \frac{(4k+1)(k+1)}{(4k+1)(4k+5)}$ = RHS

since true for n=1
if true for n=k then
true for n=2, 3 and
so on
∴ true for all values
of n

(4)

c) i) $(4+4)+(8+8)+\ldots$

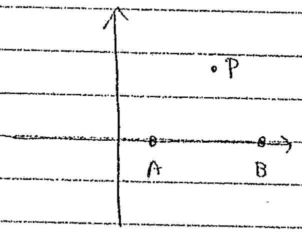
$= 2(4+8+12+\ldots 36)$

$= 2 \cdot \frac{9}{2}(4+36)$ (2)

$= 360$ m

ii) $24 + 2(28+32+36) = 216$ m (1)

d)



$m_{PA} = \frac{y-0}{x-1}$

$= \frac{y}{x-1}$

$m_{PB} = \frac{y-0}{x-5}$

$= \frac{y}{x-5}$

$m_{PA} \times m_{PB} = -1$

$\frac{y}{x-1} \times \frac{y}{x-5} = -1$ (3)

$\frac{y^2}{x^2-6x+5} = -1$

$y^2 = -x^2+6x-5$

$y^2 + x^2 - 6x = -5$

$(x-3)^2 + y^2 = -5+9$

$(x-3)^2 + y^2 = 4$

circle $c(3,0)$

$r = 2$

e) i) $A_1 = 240000(1+0.01) - M$

$= 240000(1.01) - M$

$A_2 = (240000(1.01) - M)1.01 - M$ (2)

$= 240000(1.01)^2 - 1.01M - M$

$= 240000(1.01)^2 - M(1+1.01)$

ii) $A_n = 240000(1.01)^n - M(1+1.01+\ldots 1.01^{n-1})$

$A_{240} = 240000(1.01)^{240} - M(1+1.01+\ldots 1.01^{239})$

$A_{240} = 0$

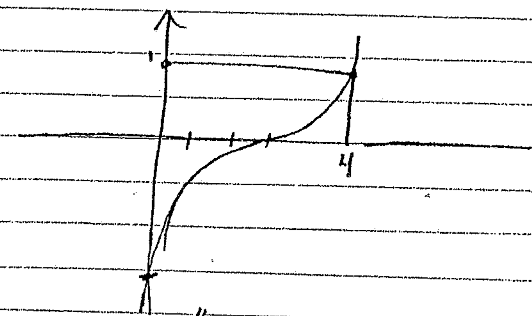
$M\left(\frac{1(1.01^{240}-1)}{0.01}\right) = 240000(1.01)^{240}$

$M = \$2642.60$ (3)

iii) $A_{60} = 240000(1.01)^{60} - 2642.60\left(\frac{1.01^{60}-1}{0.01}\right)$

$= \$220186.94$ (2)

f)



b) $x=4 \quad y=1$ (3)

$V = \pi \times 16 - \pi \int_0^1 (y^{\frac{1}{3}}+3)^2 dy$

$= 16\pi - \pi \int_0^1 y^{\frac{2}{3}} + 6y^{\frac{1}{3}} + 9 \, dy$

$= 16\pi - \left[\frac{y^{\frac{5}{3}}}{\frac{5}{3}} + \frac{6y^{\frac{4}{3}}}{\frac{4}{3}} + 9y\right]_0^1$

$= 1\frac{9}{10}\pi$

a) $V = \pi \int_3^4 ((x-3)^3)^2 dx$

$= \pi \left[\frac{(x-3)^7}{7}\right]_3^4 = \pi\left[\frac{1}{7} - 0\right]$ (2)