FORM V MATHEMATICS & EXTENSION 1

Time allowed: 3 hours

Exam date: 16th October 2003

Instructions:

All questions may be attempted.
All questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.

Collection:

Start each question in a new writing booklet.
If you don't attempt a question, hand in a blank booklet with name and class on it.
If you use a second booklet for a question, place it inside the first. Don't staple.
Write your name, class and master's initials on each writing booklet:

5A: WMP	5B: GJ	5C: JCM	5D: REP
5E: TCW	5F: MLS	5G: DS	5H: KWM

Checklist:

Folded A4 writing booklets required — 8 per boy.
Candidature: 136 boys

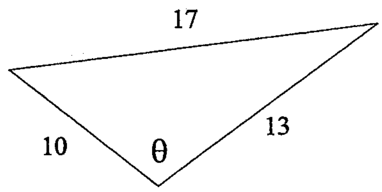
QUESTION ONE (Start a new writing booklet)

(a) Expand $(\sqrt{5}+2)^2$.

(b) Factorise $x^3 - 64$.

(c) Given the points $A(-7,-4)$ and $B(-2,11)$, find the coordinates of the point P that divides the interval AB in the ratio $3:2$.

(d)



Find θ, correct to the nearest minute, in the diagram above.

(e) (i) Sketch the parabola $y = x^2 - 5$, showing the x and y intercepts.

(ii) Hence, or otherwise, solve the inequation $x^2 < 5$.

(f) What is the gradient of the line $5x + 8y + 3 = 0$?

(g) Solve the equation $\sin\alpha = -\dfrac{1}{\sqrt{2}}$, for $0^\circ \le \alpha \le 360^\circ$.

(h) The function $g(x)$ is defined by the rule $g(x) = x + 2$.

(i) Find the value of x for which $g(x) = -1$.

(ii) Find $g(x+2)$.

QUESTION TWO (Start a new writing booklet)

(a) Differentiate each of the following functions:

(i) $y = x^5 + 5x^2 - 6x + 7$

(ii) $y = \dfrac{2}{x}$

(iii) $y = (3x - 1)^5$

(b) Find $\displaystyle\int \frac{2}{\sqrt{x}}\,dx$.

(c) Evaluate $\displaystyle\int_1^3 (2x^2 + 1)\,dx$.

(d) Consider the sequence 721, 708, 695, ...

(i) Show that the sequence is arithmetic.

(ii) Show that $T_n = 734 - 13n$, where T_n is the nth term of the sequence.

(iii) Determine whether or not -5246 is a term of the sequence.

(iv) Find the first negative term of the sequence.

(e) Find the sum of the first eight terms of the geometric series $\frac{3}{8} - \frac{1}{4} + \frac{1}{6} - \cdots$.
(Give your answer as a fraction in lowest terms.)

QUESTION THREE (Start a new writing booklet)

(a) The parabola $\mathcal{P}$ has equation $y^2 + 12x = 0$.

(i) Write down the equation of the axis of symmetry of $\mathcal{P}$.

(ii) Find the coordinates of the focus of $\mathcal{P}$.

(iii) Find the equation of the directrix of $\mathcal{P}$.

(iv) What is the length of the latus rectum of $\mathcal{P}$? (The latus rectum is the focal chord parallel to the directrix.)

(b) Solve the inequation $\dfrac{2}{x-3} < 1$.

(c) Evaluate $\displaystyle\lim_{x \to -1} \frac{x^2 - 2x - 3}{2x^2 - x - 3}$.

(d) Find, in general form, the equation of the normal to the curve $y = x^3 + x$ at the point on the curve where $x = -1$.

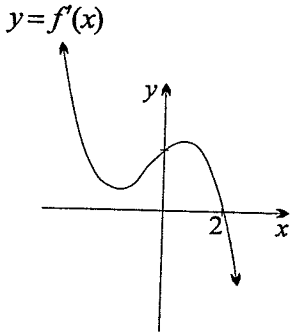
(e) (i) Calculate the discriminant of the quadratic expression $144x^2 - 470x + 161$.

(ii) Hence state the number of real roots of the equation $144x^2 - 470x + 161 = 0$.

(iii) Would it be possible to factorise the expression $144x^2 - 470x + 161$ using only rational coefficients? Give a reason for your answer.

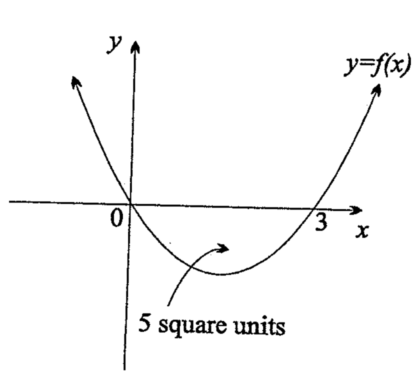
QUESTION FOUR (Start a new writing booklet)

(a)



In the above diagram, the gradient function $y = f'(x)$ of the function $y = f(x)$ is graphed. What type of stationary point does the curve $y = f(x)$ have at the point where $x = 2$?

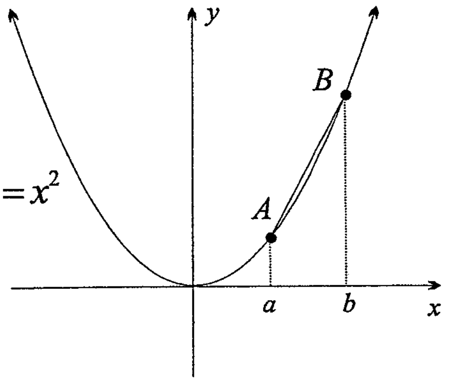
(b)



The curve $y = f(x)$ is graphed in the diagram above. The area bounded by the curve and the x-axis is 5 square units. Write down the value of $\int_0^3 f(x)\,dx$.

QUESTION FOUR (Continued)

(c)



In the diagram above, A and B are points on the parabola $y = x^2$. The x-coordinate of A is a, and the x-coordinate of B is b.

(i) Find the gradient of the chord AB in terms of a and b.

(ii) What is the limiting value of the gradient of the chord AB as b approaches a?

(iii) What is the geometrical significance of this limiting value?

(d) (i) Use the product rule to show that the derivative of the function $y = x^2(x-2)^4$ is $\dfrac{dy}{dx} = 2x(3x-2)(x-2)^3$.

(ii) Hence find the value of $\displaystyle\int_0^3 x(3x-2)(x-2)^3\,dx$.

(e) The line $y = 4x + k$ is a tangent to the parabola $x^2 = 2(y-2)$.

(i) By solving simultaneously, show that $x^2 - 8x + (4 - 2k) = 0$.

(ii) Find the discriminant of the quadratic equation in part (i).

(iii) Hence find the value of k.

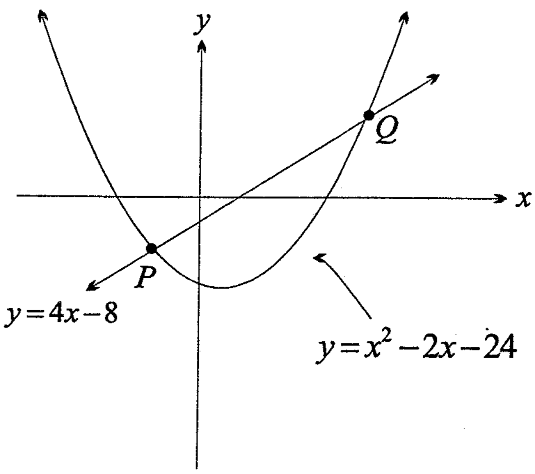
QUESTION FIVE (Start a new writing booklet)

(a) Find $\int (2x+1)^5 \, dx$.

(b) (i) Use Simpson's rule with five function values to approximate $\int_2^6 \log_{10} x \, dx$.
(Give your answer correct to six decimal places.)

(ii) If the exact value of the integral in part (i) is $2{\cdot}32966958\ldots$, to how many decimal places is your approximation accurate?

(c)



The diagram above shows the parabola $y = x^2 - 2x - 24$ and the line $y = 4x - 8$.

(i) By solving simultaneously, show that the points of intersection of the line and the parabola are $P(-2,-16)$ and $Q(8,24)$.

(ii) Calculate the area enclosed between the line and the parabola.

(d) The curve $y = Ax^3 + Bx^2 + Cx$ has a point of inflexion at $x = \frac{1}{2}$, a stationary point at $x = -1$ and passes through the point $(1, 13)$. Find A, B and C.

QUESTION SIX (Start a new writing booklet)

Consider the curve with equation $y = \dfrac{5x^2 - 18x + 45}{x^2 - 9}$.

(a) Show that there are no x-intercepts.

(b) Find the y-intercept.

(c) Write down the equations of the vertical asymptotes.

(d) Show that $(1, -4)$ and $(9, 4)$ are stationary points, and determine their nature.

(e) Find the equation of the horizontal asymptote.

(f) Find the point where the curve crosses the horizontal asymptote.

(g) Sketch the curve showing all the features found in the previous parts.

(h) State the range of the function $y = \dfrac{5x^2 - 18x + 45}{x^2 - 9}$.

(i) Show that the equation of the curve can be written as

$$(5 - y)x^2 - 18x + 9(5 + y) = 0.$$

(j) Use the discriminant of the quadratic equation in part (i) to confirm algebraically the range of the function found in part (h).

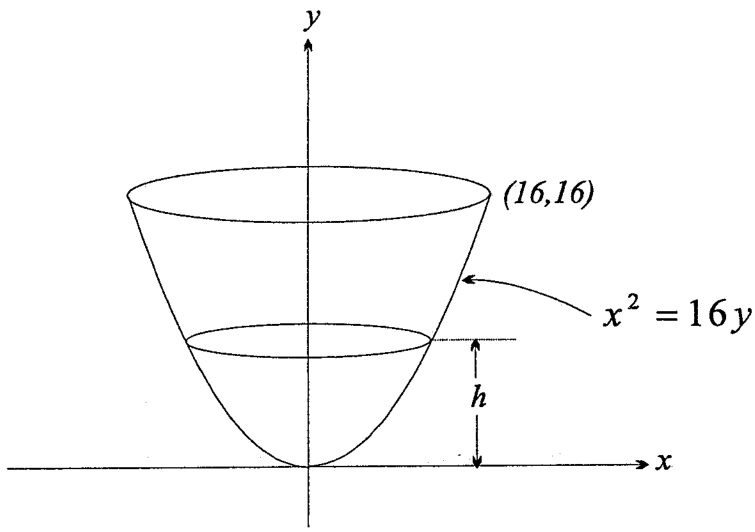
Exam continues next page ...

QUESTION SEVEN (Start a new writing booklet)

(a) The variable point $P(x, y)$ moves in such a way that it is always twice as far from the line $x + y = 1$ as it is from the point $A(1, 1)$. Show that the locus of P has equation

$$7(x-1)^2 + 7(y-1)^2 + 1 = 2xy.$$

(b)



The diagram above (which is not drawn to scale) shows a parabolic bowl that is formed by rotating the parabola $x^2 = 16y$ from $(0, 0)$ to $(16, 16)$ about the y-axis.

(i) Some water is poured into the bowl. If the greatest depth of the water is h cm, as shown in the diagram, show that the volume of water in the bowl is $8\pi h^2$ cm^3.

(ii) The water was poured into the bowl at a constant rate of 120 cm^3 per second. Find the rate at which h was increasing when h was 6 cm.

(c) Consider the quadratic equation $x^2 - 5kx + 9k = 0$, where k is a constant, and let α and β be its two real roots.

(i) Write down expressions for $\alpha + \beta$ and $\alpha\beta$ in terms of k.

(ii) Find the possible values of k if α and β differ by 16.

QUESTION EIGHT (Start a new writing booklet)

(a) (i) Find the sum of the series $1 + 2 + 3 + \cdots + n$.

(ii) Prove, by mathematical induction, that for all positive integer values of n

$$(1^5 + 2^5 + 3^5 + \cdots + n^5) + (1^7 + 2^7 + 3^7 + \cdots + n^7) = 2(1 + 2 + 3 + \cdots + n)^4.$$

(b) Consider the parabola with parametric equations $x = 2at$ and $y = at^2$.

(i) Derive the equation of the tangent to the parabola at the variable point $(2at, at^2)$.

(ii) P is the point on the parabola whose parameter is $t = p + \dfrac{1}{p}$, for $p \neq 0$, and Q is the point on the parabola whose parameter is $t = p - \dfrac{1}{p}$.

The tangents to the parabola at P and Q intersect at T. Show that the locus of T, as p varies, has Cartesian equation

$$y = a\left(\frac{x^2}{4a^2} - \frac{4a^2}{x^2}\right).$$

(c) A boat A leaves port P and sails at a constant speed in a straight line due north.

At the same time, a second boat B is sailing in a straight line towards P, travelling in the direction S60°W. Boat B is sailing at twice the speed of boat A.

Find the ratio $AP : BP$ of the distances of the two boats from port P at the instant when the two boats are nearest one another.

DS

SOLUTIONS TO F5 Ext 1 EXAM, 2003

Total is $8 \times 15 = 120$

(1) (a) $9 + 4\sqrt{5}$

(b) $(x-4)(x^2+4x+16)$ (-1 per error)

(c) $P = \left(\frac{3x-2+2x-7}{3+2}, \frac{3x11+2x-4}{3+2}\right)$

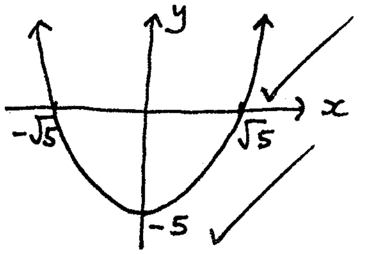
$= (-4, 5)$ (-1 per error)

(d) $\cos\theta = \frac{10^2+13^2-17^2}{2 \times 10 \times 13}$

$= -\frac{1}{13}$

$\therefore \theta = 94°25'$ (accept 94° or 94°24')

(e)(i)



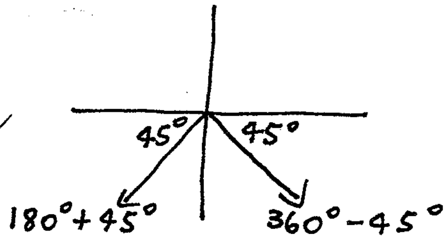
(ii) $-\sqrt{5} < x < \sqrt{5}$

(f) $y = -\frac{5}{8}x - \frac{3}{8}$

$\therefore m = -\frac{5}{8}$

(g)

$\alpha = 225°$ or $315°$



(h)(i) $x + 2 = -1$

$\therefore x = -3$

(ii) $g(x+2) = (x+2)+2$

$= x + 4$

(2) (a)(i) $y' = 5x^4 + 10x - 6$ ✓

(ii) $y = 2x^{-1}$ ✓

$$\left.\begin{aligned} y' &= -2x^{-2} \\ &= \frac{-2}{x^2} \end{aligned}\right\} \checkmark \text{(either)}$$

(iii) $y' = 5(3x-1)^4 . 3$ ✓

$= 15(3x-1)^4$ ✓

(e) $a = \frac{3}{8}$, $r = -\frac{2}{3}$, $n = 8$

$$S_8 = \frac{\frac{3}{8}\left[\left(-\frac{2}{3}\right)^8 - 1\right]}{-\frac{2}{3} - 1} \checkmark$$

$$\checkmark = \frac{3}{8} \times \frac{3}{5} \times \left(1 - \frac{256}{6561}\right)$$

$$= \frac{9}{40} \times \frac{6305}{6561}$$

$$\checkmark = \frac{1261}{5832}$$

(b) $\int 2x^{-\frac{1}{2}} dx = \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$ ✓ ✓

$= 4\sqrt{x} + c$ ✓

(no penalty if "c" omitted)

(c) $\int_1^3 (2x^2+1)dx = \left[\frac{2x^3}{3} + x\right]_1^3$ ✓

$= (18 + 3) - \left(\frac{2}{3} + 1\right)$

$= 19\frac{1}{3}$ ✓

(d)(i)

$$\left.\begin{aligned} T_3 - T_2 &= 695 - 708 \\ &= -13 \\ \text{and } T_2 - T_1 &= 708 - 721 \\ &= -13 \end{aligned}\right\} \checkmark$$

$\therefore$ the sequence is arithmetic.

(ii)

$$\left.\begin{aligned} T_n &= a + (n-1)d \\ &= 721 - 13(n-1) \\ &= 734 - 13n \end{aligned}\right\} \checkmark$$

(iii) Let $T_n = -5246$ and solve for n:

$$\left.\begin{aligned} -5246 &= 734 - 13n \\ 13n &= 5980 \\ n &= 460, \end{aligned}\right\} \checkmark$$

so -5246 is T_{460}.

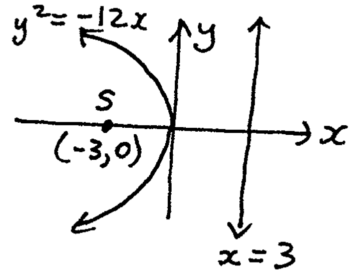
(iv) Set $T_n < 0$ and solve for n:

$$\left.\begin{aligned} 734 - 13n &< 0 \\ 13n &> 734 \\ n &> 56.46\ldots \end{aligned}\right\} \checkmark$$

So $T_{57} = 721 - \cancel{13(57)}\ 13 \times 56$ (or $734 - 13 \times 57$)

$= -7$ is the first negative term.

(3) (a)(i) $y = 0$



(ii) $4a = 12$

$\therefore a = +3$

$\therefore S = (-3, 0)$

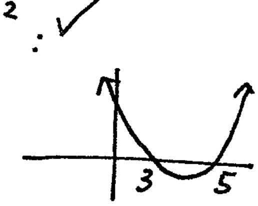
(iii) $x = 3$

(iv) $4a = 12$ units

(b) Multiply both sides by $(x-3)^2$:

$2(x-3) < (x-3)^2$

$(x-3)(x-5) > 0$



$x < 3$ or $x > 5$

(c) $\lim_{x \to -1} \frac{(x-3)(x+1)}{(2x-3)(x+1)} = \lim_{x \to -1} \frac{x-3}{2x-3}$

$= \frac{-4}{-5}$

$= \frac{4}{5}$

(d) When $x = -1$, $y = -1 - 1$

$= -2$

$y' = 3x^2 + 1$

At $(-1, -2)$, gradient of tangent $= 3(-1)^2 + 1$

$= 4$.

So the gradient of the normal is $-\frac{1}{4}$.

The eqn of the normal is

$y + 2 = -\frac{1}{4}(x+1)$

$4y + 8 = -x - 1$

$x + 4y + 9 = 0$

(e)(i) $\Delta = (-470)^2 - 4(144)(161)$

$= 128\,164$

(ii) Since $\Delta > 0$, there are <u>two</u> real roots.

(iii) $128164 = 358^2$, so it <u>would</u> be possible.

(4) (a)

x	2^-	2	2^+
$f'(x)$	$+$	0	$-$
	/	—	\

There is a maximum turning point at $x = 2$ on the curve $y = f(x)$.

(b) $\int_0^3 f(x)\,dx = -5$

(c) (i) $m_{AB} = \dfrac{b^2 - a^2}{b - a}$

$= b + a$

(ii) $a + a = 2a$

(iii) This limiting value is the gradient of the tangent to the parabola at A.

(d)(i) $y = x^2(x-2)^4$

Let $u = x^2$
$\therefore u' = 2x$
Let $v = (x-2)^4$
$\therefore v' = 4(x-2)^3$

$y' = vu' + uv'$

$= 2x(x-2)^4 + 4x^2(x-2)^3$

$= 2x(x-2)^3\left[(x-2) + 2x\right]$

$= 2x(x-2)^3(3x-2)$

(ii) $\frac{1}{2}\int_0^3 2x(3x-2)(x-2)^3\,dx = \frac{1}{2}\left[x^2(x-2)^4\right]_0^3$

$= \frac{1}{2}(9 - 0)$

$= 4\frac{1}{2}$

(e) (i) Substitute $y = 4x + k$ into $x^2 = 2(y-2)$.

$\therefore x^2 = 2(4x + k - 2)$

$x^2 = 8x + 2k - 4$

$x^2 - 8x + (4 - 2k) = 0$

(ii) $\Delta = (-8)^2 - 4(1)(4 - 2k)$

$= 64 - 16 + 8k$

$= 48 + 8k$

(iii) Since the line is a tangent, $\Delta = 0$.

$\therefore 48 + 8k = 0$

$\therefore k = -6$

(5)(a) $\int (2x+1)^5\,dx = \dfrac{(2x+1)^6}{12} + c$ ✓ (no penalty for omitting "c")

(b)(i) $\int_2^6 \log_{10} x\,dx \doteqdot \frac{1}{3}\left(\log_{10}2 + 4\log_{10}3 + 2\log_{10}4 + 4\log_{10}5 + \log_{10}6\right)$ ✓✓ (−1 per error)

$\doteqdot 2\cdot329222$ ✓

(ii) It is accurate to <u>two</u> decimal places. ✓
(Accept three.)

(c)(i) $y = 4x - 8$ ①

$y = x^2 - 2x - 24$ ②

Substitute from ② into ①:

$$\begin{aligned}\therefore\ x^2 - 2x - 24 &= 4x - 8\\ \therefore\ x^2 - 6x - 16 &= 0\\ (x+2)(x-8) &= 0\\ x &= -2 \text{ or } 8\end{aligned}$$ ✓

When $x = -2$,

$$\begin{aligned}y &= 4(-2) - 8\\ &= -12\end{aligned}$$

When $x = 8$,

$$\begin{aligned}y &= 4(8) - 8\\ &= 24\end{aligned}$$ ✓

So the points of intersection are $(-2,-12)$ and $(8,24)$.

(ii) Area $= \int_{-2}^{8}\left[4x - 8 - (x^2 - 2x - 24)\right]dx$ ✓

$$\begin{aligned}&= \int_{-2}^{8}(6x + 16 - x^2)\,dx\\ &= \left[3x^2 + 16x - \frac{x^3}{3}\right]_{-2}^{8} \checkmark\\ &= \left(192 + 128 - \frac{512}{3}\right) - \left(12 - 32 + \frac{8}{3}\right)\\ &= \frac{500}{3} \text{ or } 166\tfrac{2}{3} \text{ square units}\end{aligned}$$ ✓

(d) $y' = 3Ax^2 + 2Bx + C$

$y'' = 6Ax + 2B$

$3A + 2B = 0$ (1)

$3A - 2B + C = 0$ (2)

$A + B + C = 13$ (3)

(1) + (2) : $6A + C = 0$ (4)

(2) + 2×(3) : $5A + 3C = 26$ (5)

From (4), $C = -6A$

Substitute into (5):

$5A - 18A = 26$

$-13A = 26$

$\therefore A = -2,\ B = 3,\ C = 12$

(6) (a) Let $y = 0$

$\therefore \quad 5x^2 - 18x + 45 = 0$

$\Delta = 18^2 - 20 \times 45 < 0$

So there are no x-intercepts.

(b) Let $x = 0$

$\therefore \quad y = -5$

(d) By the quotient rule,

$$y' = \frac{(x^2-9)(10x-18) - 2x(5x^2-18x+45)}{(x^2-9)^2}$$

$$= \frac{10x^3 - 18x^2 - 90x + 162 - 10x^3 + 36x^2 - 90x}{(x^2-9)^2}$$

$$= \frac{18x^2 - 180x + 162}{(x^2-9)^2}$$

$$= \frac{18(x^2 - 10x + 9)}{(x^2-9)^2}$$

Let $y' = 0$ for stationary points.

$\therefore \quad x^2 - 10x + 9 = 0$

$(x-1)(x-9) = 0$

$x = 1$ or 9

When $x = 1$, $y = -4$.
When $x = 9$, $y = 4$.

x	0	1	2	8	9	10
y'	2	0	-5.04	$\frac{-126}{3025}$	0	$\frac{162}{8281}$
	/	−	\	\	−	/

$(1, -4)$ is a maximum turning point.
$(9, 4)$ " " minimum " "

(c) $x = -3$, $x = 3$

(e) As $x \to \pm\infty$,

$y \to 5$.

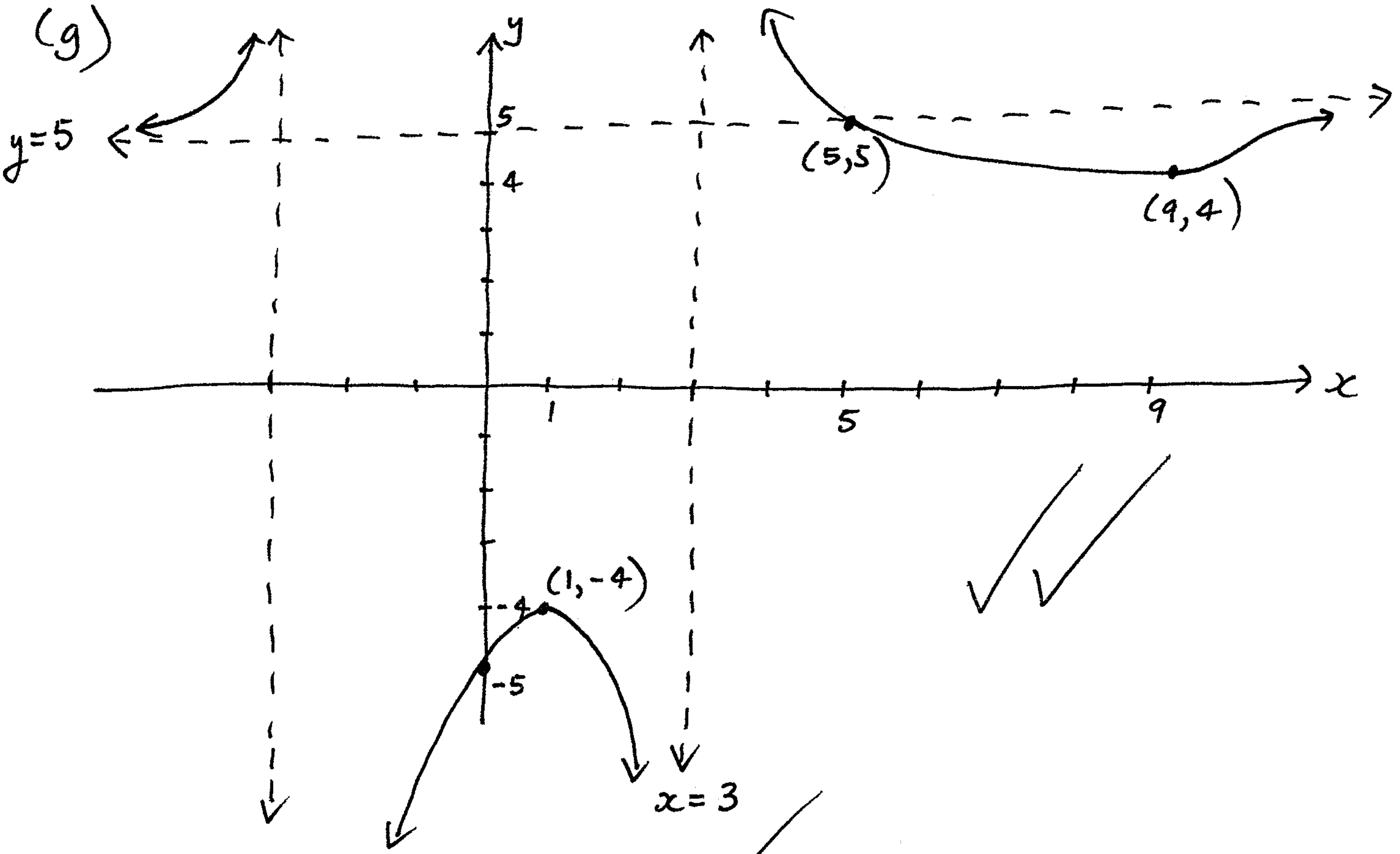
So $y = 5$ is a horizontal asymptote.

(f) Let $y = 5$.

$$\therefore\ 5x^2 - 45 = 5x^2 - 18x + 45$$
$$18x = 90$$
$$x = 5.$$

So the point is $(5, 5)$. ✓

(g)



✓✓

✓

(h) $y \leq -4$ or $y \geq 4$ ✓

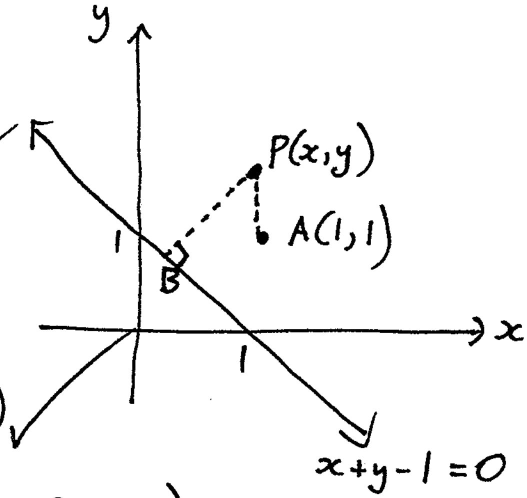
(i) $yx^2 - 9y = 5x^2 - 18x + 45$ ✓

$\therefore\ (5-y)x^2 - 18x + (45+9y) = 0$ ✓

(j)
$$\begin{aligned}\Delta &= 18^2 - 4(5-y)(45+9y) \\ &= 18^2 - 36(25-y^2) \\ &= 36(9 - 25 + y^2) \\ &= 36(y^2 - 16)\end{aligned}$$
✓

For real values of x, we require $\Delta \geq 0$,
so $y^2 - 16 \geq 0$,
so $y \leq -4$ or $y \geq 4$. ✓

(7) (a)



$PB = 2 \cdot PA$

$\therefore \left| \frac{x+y-1}{\sqrt{2}} \right| = 2\sqrt{(x-1)^2+(y-1)^2}$

$\therefore \frac{(x+y-1)^2}{2} = 4(x^2-2x+y^2-2y+2)$

$\therefore (x+y)^2 - 2(x+y) + 1 = 8(x^2-2x+y^2-2y+2)$

$x^2+2xy+y^2-2x-2y+1 = 8x^2-16x+8y^2-16y+16$

$2xy = 7x^2-14x+7y^2-14y+15$

$\therefore 2xy = 7(x^2-2x+1) + 7(y^2-2y+1) + 1$

$\therefore 7(x-1)^2 + 7(y-1)^2 + 1 = 2xy$

(b)(i) $V = \pi \int_0^h 16y\, dy$

$= \pi [8y^2]_0^h$

$= 8\pi h^2 \text{ cm}^3$

(ii) $V = 8\pi h^2$

$\therefore \frac{dV}{dt} = 16\pi h \cdot \frac{dh}{dt}$

$120 = 16\pi h \cdot \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{15}{2\pi h}$

When $h = 6$,

$\frac{dh}{dt} = \frac{15}{12\pi}$

$= \frac{5}{4\pi}$ cm/s

So at the instant when $h = 6$ cm,
h is increasing at $\frac{5}{4\pi}$ ($\doteqdot 0.4$) cm/s.

(c) (i) $\alpha + \beta = 5k$ ✓

$\alpha\beta = 9k$ ✓

(ii) $\alpha + \beta = 5k$ (1)

$\alpha\beta = 9k$ (2)

Suppose, without loss of generality, that $\alpha > \beta$.

$\therefore \alpha - \beta = 16$ (3)

From (3), $\alpha = 16 + \beta$

Substitute into (1) and (2):

$16 + 2\beta = 5k$ (4)

$\beta(16 + \beta) = 9k$ (5) ✓

From (4), $\beta = \dfrac{5k - 16}{2}$

Substitute into (5):

$$\frac{5k-16}{2} \cdot \frac{5k+16}{2} = 9k$$ ✓

$$25k^2 - 256 = 36k$$

$$25k^2 - 36k - 256 = 0$$ ✓

$$(k-4)(25k+64) = 0$$

$$\therefore k = 4 \text{ or } k = \frac{-64}{25}$$ ✓

OR Square both sides of (3):

$$\alpha^2 + \beta^2 - 2\alpha\beta = 16^2$$

$$(\alpha+\beta)^2 - 4\alpha\beta = 256$$

Using (1) and (2):

$$(5k)^2 - 4(9k) = 256$$

$$25k^2 - 36k - 256 = 0$$

etc

(8) (a) (i) $\frac{1}{2}n(n+1)$ ✓

(ii) When $n=1$,

$$LHS = 1^5 + 1^7 = 2$$
$$RHS = 2(1)^4 = 2$$

So the result is true for $n=1$. ✓

Suppose that the result is true for the positive integer $n=k$.

i.e. suppose that

$$(1^5 + 2^5 + \ldots + k^5) + (1^7 + 2^7 + \ldots + k^7) = 2(1+2+\ldots+k)^4$$
$$= 2\left[\tfrac{1}{2}k(k+1)\right]^4$$
$$= \tfrac{1}{8}k^4(k+1)^4 \quad (*)$$

Prove that the result is true for $n=k+1$.

i.e. prove that

$$(1^5 + 2^5 + \ldots + k^5 + (k+1)^5) + (1^7 + 2^7 + \ldots + k^7 + (k+1)^7) = \tfrac{1}{8}(k+1)^4(k+2)^4$$ ✓

$$LHS = (k+1)^5 + (k+1)^7 + \tfrac{1}{8}k^4(k+1)^4 \quad [\text{by } (*)]$$
$$= \tfrac{1}{8}(k+1)^4\left[8(k+1) + 8(k+1)^3 + k^4\right]$$
$$= \tfrac{1}{8}(k+1)^4(k^4 + 8k + 8 + 8k^3 + 24k^2 + 24k + 8)$$
$$= \tfrac{1}{8}(k+1)^4(k^4 + 8k^3 + 24k^2 + 32k + 16)$$ ✓

Now, $(k+2)^4 = (k+2)^2(k+2)^2$

$$= (k^2+4k+4)(k^2+4k+4)$$
$$= k^4 + 4k^3 + 4k^2 + 4k^3 + 16k^2 + 16k + 4k^2 + 16k + 16$$
$$= k^4 + 8k^3 + 24k^2 + 32k + 16$$

$$\therefore LHS = \tfrac{1}{8}(k+1)^4(k+2)^4$$
$$= RHS$$ ✓

So the result is true for $n=k+1$ if it is true for $n=k$. But the result is true for $n=1$. So, by induction, the result is true for all positive integer values of n.

(b) (i) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ or $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$= \frac{2at}{2a}$ $\quad \frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$

$= t$ ✓ $\quad$ At $x = 2at$, $\frac{dy}{dx} = \frac{2at}{2a}$

$= t$

The equation of the tangent is

$$\left.\begin{aligned} y - at^2 &= t(x - 2at) \\ y - at^2 &= tx - 2at^2 \\ y &= tx - at^2 \end{aligned}\right\} ✓$$

(ii) Tangent at P : ✓ $y = \left(p + \frac{1}{p}\right)x - a\left(p + \frac{1}{p}\right)^2$ (1)

Tangent at Q : $y = \left(p - \frac{1}{p}\right)x - a\left(p - \frac{1}{p}\right)^2$ (2)

Find T :

$\therefore \left(p + \frac{1}{p}\right)x - a\left(p^2 + 2 + \frac{1}{p^2}\right) = \left(p - \frac{1}{p}\right)x - a\left(p^2 - 2 + \frac{1}{p^2}\right)$

$\frac{2}{p}x = 4a$

$\therefore x = 2ap$ (3)

Sub. into (1): $y = \left(p + \frac{1}{p}\right)2ap - a\left(p^2 + 2 + \frac{1}{p^2}\right)$

$= 2ap^2 + 2a - ap^2 - 2a - \frac{a}{p^2}$

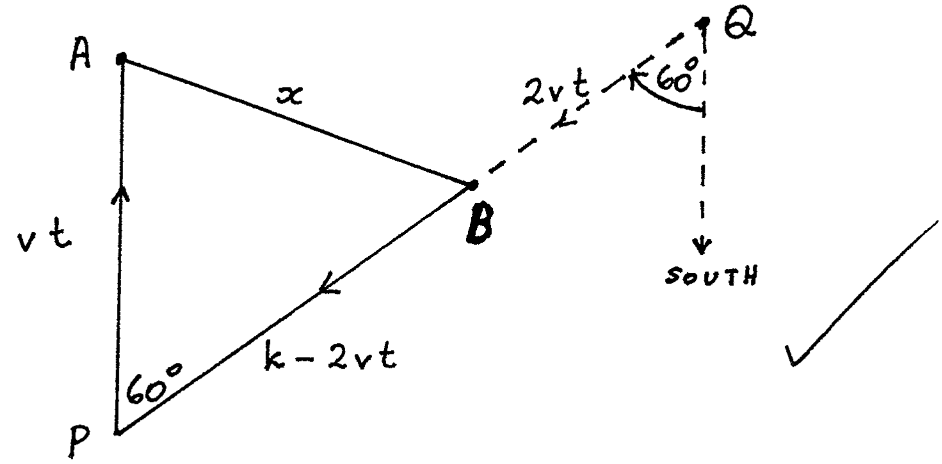
$\therefore y = ap^2 - \frac{a}{p^2}$ (4) ✓

From (3), $p = \frac{x}{2a}$

Sub. into (4) : $y = a \cdot \frac{x^2}{4a^2} - a \cdot \frac{4a^2}{x^2}$

$y = a\left(\frac{x^2}{4a^2} - \frac{4a^2}{x^2}\right)$ ✓

(c) Suppose that ship B was at point Q when ship A was just leaving port P.



Let v be the speed of boat A, so that the speed of boat B is 2v. (v is constant)

Let t be the time elapsed since boat A left port P. So t is also the time elapsed since boat B was at Q. (t varies)

Let AB = x, and let QP = k. (x varies, k is constant)

By the cosine rule,

$$x^2 = (vt)^2 + (k-2vt)^2 - 2(k-2vt)(vt)\cos 60°$$
$$= v^2t^2 + k^2 - 4kvt + 4v^2t^2 - kvt + 2v^2t^2$$
$$= 7v^2t^2 - 5kvt + k^2$$

$$\therefore \frac{d}{dt}(x^2) = 14v^2t - 5kv$$
$$= 0 \text{ when } t = \frac{5k}{14v}$$

$$\frac{d^2}{dt^2}(x^2) = 14v^2 > 0 \text{ for all } t,$$

so $t = \frac{5k}{14v}$ minimises x^2, and hence x.

when $t = \frac{5k}{14v}$,

$$AP = \frac{5k}{14} \text{ and } BP = k - \frac{5k}{7} = \frac{2k}{7}$$

$$\therefore AP : BP = \frac{5k}{14} : \frac{2k}{7} = 5 : 4$$